Quadratic funktions

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1 Preface

To create instructions like this it takes a lot of time and effort. Nevertheless, I provide this to the community free of charge. If you find this script helpful, I request you to fulfill the inter-generation contract, describes as below:

> *When you some time later will have finished your education and you have started your career (or later), please transmit your knowledge in a fitting way to the following generation.*

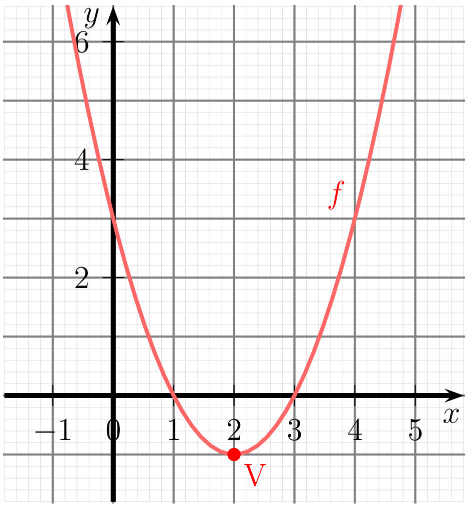
If you want to make me happy, please send a small email to the following address: mail@dk4ek.de

Many thanks!

2 Compilation of fundamental facts

On the right you will find a diagramm of a typical quadratic funktion. The example represents this function:

$$f(x) = x^2 - 4x + 3$$



This curve shape is called **parabola**. The lowest point[1] is called **vertex of the parabola**. This point is usually indicated with the letter **V**.

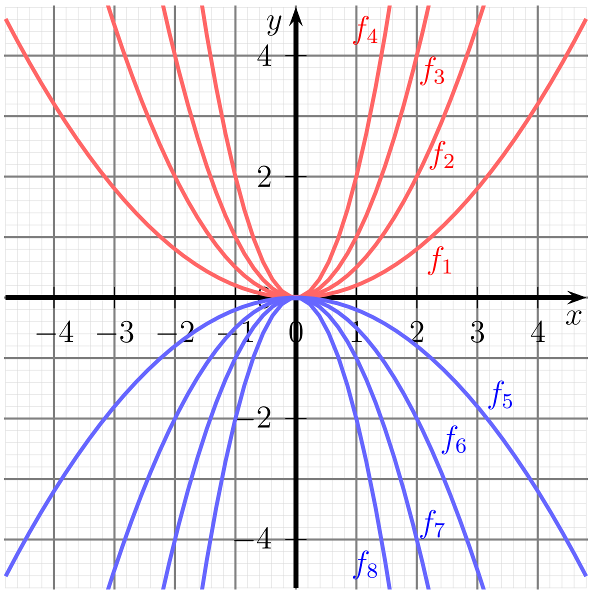
A quadratic function is any function, that fits in the normal form:

normal form:

$$f(x) = ax^2 + bx + c$$

In order to analyse, what kind of effect the parameter a has to the shape of the parabola, you can see below 8 different functions with a different value of the parameter a. In all functions is $b = 0$ and $c = 0$. The functional equations are:

$f_1(x) = 0{,}2x^2$
$f_2(x) = 0{,}5x^2$
$f_3(x) = 1x^2$
$f_4(x) = 2x^2$
$f_5(x) = -0{,}2x^2$
$f_6(x) = -0{,}5x^2$
$f_7(x) = -1x^2$
$f_8(x) = -2x^2$

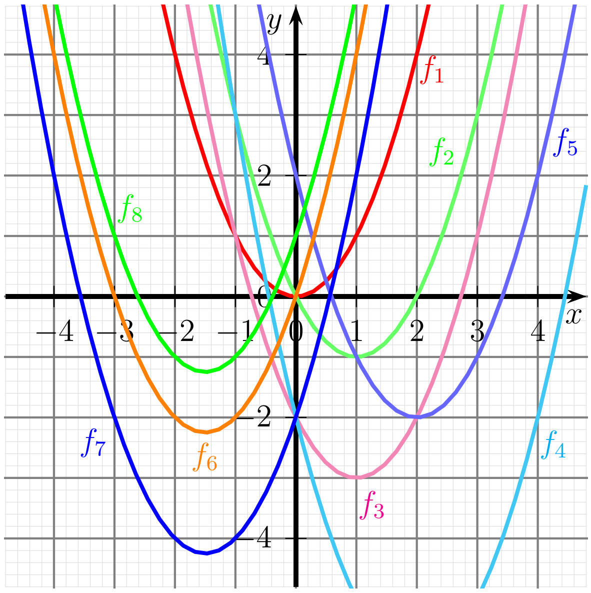


It is easy to see, that all parabolas have a different shape. As the parameter $\boldsymbol{a}$ is determining the shape of the parabola, it is called **factor of shape**. If the absolute value of $\boldsymbol{a}$ is **large** (f_3, f_4, f_7, f_8), the parabola is **narrow**, if $\boldsymbol{a}$ is **negative** (f_5 to f_8), the parabola is **opened downwards**.

[1] If the parabola is opened downwards, the vertex is the highest point.

To demonstrate, that only parameter $\boldsymbol{a}$ is responsible for the shape of the parabola, you see right 8 differert quadratic functions. All have parameter $a = 1$, but different values for parameter b and c. The functional equations are:

$f_1(x) = x^2$
$f_2(x) = x^2 - 2x$
$f_3(x) = x^2 - 2x - 2$
$f_4(x) = x^2 - 4x - 2$
$f_5(x) = x^2 - 4x + 2$
$f_6(x) = x^2 + 3x$
$f_7(x) = x^2 + 3x - 2$
$f_8(x) = x^2 + 3x + 1$



Although you see a lot of confused lines, you can recognize, that all parabolas have the same shape. Only their positions are different.

Next we want to explore the impact of parameter $\boldsymbol{c}$. We can use the same functions f_1 to f_8 shown before for this purpose.

Let us look first to the functions f_1, f_2 and f_6. In all of them is $c = 0$. You should notice, that all parabolas pass through the coordinate origin. Here is $y = 0$.

Also the functions f_3, f_4 and f_7 have the same value for parameter c, namely $c = -2$. All associated parabulas are hitting the y-axis at the same value, namely at $y = -2$.

Obviously the parameter $\boldsymbol{c}$ gives us the value for the $\boldsymbol{y}$**-intercept**.

We can verify this finding with f_5 und f_8. In f_5 is $c = 2$, the y-axis is hitted at $y = 2$. In f_8 is $c = 1$, the y-axis is hitted at $y = 1$. Obviously it is truth:

Parameter $\boldsymbol{c}$ indicates the $\boldsymbol{y}$**-intercept**.

Note: Of course ist is possible to prove this result "correctly". When you calculate the value of the function $f(x) = ax^2 + bx + c$ with $x = 0$, you get $a \cdot 0^2 = 0$ and $b \cdot 0 = 0$. Then only the value of c is remaining.

Unfortunately the meaning of parameter b is not as evident as parameter a or c. Therefore I want to inspect this later in this script.

2.1 Zero of a function

Zero of a function is called the value x_0, that causes the value 0 of the function (the value $y = 0$). To find possible zeros of a function, we have to set the formula term to 0. I want to show the procedure with the example $f(x) = x^2 - 4x + 3$ from the beginning. We need the p-q-formula[2] as a tool to solve the equation.

$$\begin{aligned}
f(x_0) &= 0 \\
x_0^2 - 4x_0 + 3 &= 0 \qquad | \text{ use } p\text{-}q\text{-formula} \\
x_{01/02} &= -\frac{-4}{2} \pm \sqrt{\left(\frac{-4}{2}\right)^2 - 3} \\
&= 2 \pm \sqrt{4-3} \\
x_{01/02} &= 2 \pm 1 \\
x_{01} = 1 & \qquad x_{02} = 3
\end{aligned}$$

Result: Zeros of the function are $x_{01} = 1$ and $x_{02} = 3$.
You can express this result also in an other way: The parabel cross the x-axis at $x_{01} = 1$ and $x_{02} = 3$.

2.2 Vertexes

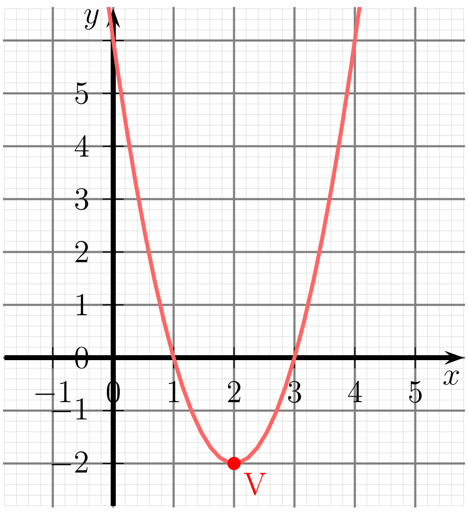
What is a **vertex of a parabula**?

The definition is as told before: *The* ***vertex*** *of a parabula is its lowest or its highest point.*

Opposite you find the graph of this function:

$$f(x) = 2x^2 - 8x + 6$$

The vertex is marked with an V. As the parabula is opened upwards, we only find a **lowest** point. There is no upward limit. Therefore is the vertex the lowest point.



Let us look to the details of the function graph. You see the vertex by reason of the symmetrie of the parabula exactly in the centre between the zeros. Because of this fact we can deduce a formula for finding the vertex. Let us first calculate the zeros of this example.

[2]For details of the p-q-formula look here: http://www.dk4ek.de/lib/exe/fetch.php/quad.pdf

$$
\begin{aligned}
f(x_0) &= 0 \\
2x_0^2 - 8x_0 + 6 &= 0 \qquad | : 2 \\
x_0^2 - 4x_0 + 3 &= 0 \\
p &= -4 \\
q &= 3 \\
x_{1/2} &= -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} \\
x_{01/02} &= -\frac{-4}{2} \pm \sqrt{\left(\frac{-4}{2}\right)^2 - 3} \\
x_{01/02} &= 2 \pm \sqrt{2^2 - 3} \\
x_{01/02} &= 2 \pm 1
\end{aligned}
$$

Without calculating the values in detail we can see by this formula, that the zeros are positioned symmetric to $x = 2$, one 1 below and one 1 above. That means, that the number **before** the sign $\pm$ must be the x-value x_V of the vertex. The vertex is positioned as we know in the center between the zeros.

Result: $\boxed{x_V = 2}$

I want to use this realization, to deduce a simple formula, that shows us the x-value of the vertex $V(x_V|y_V)$ of a quadratic function.

I start with the normal form and calculate the zeros.

$$
\begin{aligned}
f(x) &= ax^2 + bx + c \qquad | \text{ set the formula term to } 0 \\
0 &= ax^2 + bx + c \qquad | : a \\
0 &= x^2 + \frac{b}{a}x + \frac{c}{a} \qquad | \text{ use } p\text{-}q\text{-formula} \\
x_{01/02} &= -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \\
x_{01} &= -\frac{b}{2a} + \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \\
x_{02} &= -\frac{b}{2a} - \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}
\end{aligned}
$$

I calculate the mean value, to get the x-value x_V of the vertex.

$$\begin{aligned} x_V &= \frac{x_{01}+x_{02}}{2} \\ x_V &= \frac{\left(-\frac{b}{2a}+\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}}\right)+\left(-\frac{b}{2a}-\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}}\right)}{2} \\ &= \frac{-2\cdot\frac{b}{2a}}{2} \\ x_V &= -\frac{b}{2a} \end{aligned}$$

Result: If the equation of the function is given in the normal form

$$\boxed{f(x) = ax^2 + bx + c}$$

we get the x-value $\mathbf{x_V}$ of the vertex by using this formula:

vertex formula:

$$\boxed{x_V = -\frac{b}{2a}}$$

Let us go back to our example.

$$f(x) = 2x^2 - 8x + 6$$

We calculate x_V by using the formula, we just have deduced:

$$\begin{aligned} x_V &= -\frac{b}{2a} \\ &= -\frac{-8}{2\cdot 2} \\ x_V &= 2 \end{aligned}$$

The corresponding y-value y_V can be calculated by putting x_V into the function formula. Our example looks like that:

$$\begin{aligned} y_V &= f(x_V) \\ &= 2x_V^2 - 8x_V + 6 \\ &= 2\cdot 2^2 - 8\cdot 2 + 6 \\ y_V &= -2 \end{aligned}$$

We got the vertex of our example like this: $\boxed{V(2|-2)}$

2.3 vertex form

The functional equation of a quadratic function can not only be written in **normal form**, possible ist also the so called **vertex form**. We remember the normal form:

$$f(x) = ax^2 + bx + c$$

When we write the functional equation in **vertex form**, it will look like this:

vertex form:

$$\boxed{f(x) = a \cdot (x - x_V)^2 + y_V}$$

In this formula are $\mathbf{x_V}$ and $\mathbf{y_V}$ the coordinates of the vertex $V(x_V|y_V)$, parameter $\boldsymbol{a}$ is the same, that we know from the normal form as **factor of shape**.

At this point I do not want to deduce this formula. It would be possible very easy by using the following formula for shifting a function graph. If we want to shift a function graph of a arbitrary function (not only a quadratic function) by the value x_s to the right and y_s upwards, we get the functional equation of the shifted function $f_s(x)$ by using this formula:

Shifted funktion:

$$\boxed{f_s(x) = f(x - x_s) + y_s}$$

I also do not want to prove this formula at this point.[3]

Important: The vertex form provides two benefits compared to the normal form.

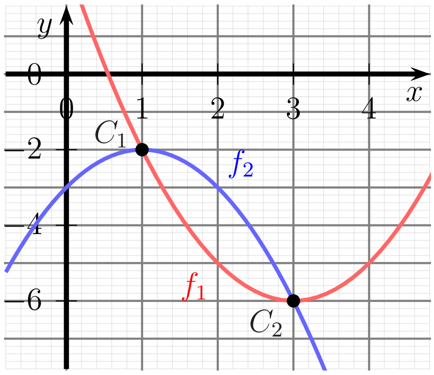
1. If the functional equation of a quadratic function is given in vertex form, we immediate can read in it the **coordinates of the vertex**.

2. When we have to find out the functional equation of a quadratic function, and we know the factor of shape a and the coordinates of the vertex, we immediately can write down the functional equation.

[3]I wrote down details of shifting function graphs here:
http://www.dk4ek.de/lib/exe/fetch.php/transfo.pdf

2.4 Determining crossing points

Also parabulas can have crossing points with each other. Opposite you see the function graphs of these functions:



$$f_1(x) = x^2 - 6x + 3 \quad \text{and} \quad f_2(x) = -x^2 + 2x - 3$$

In this case we have **two crossing points**, C_1 and C_2. It is also possible, that the parabolas have a position, that they do not hit each other. For example we imagin, that the function graph of f_1 would have a position 5 units higher. In this case the function graphs would run past **without any hit**. It is also possible, that the function graphs just touch each other with only **one common point**.

As we did it with linear functions[4] we can find out crossing points by equating the formula terms. I want to show, how it can be done by using this example.

$$\begin{aligned}
f_1(x_V) &= f_2(x_V) \\
x_V^2 - 6x_V + 3 &= -x_V^2 + 2x_V - 3 && \mid + x_V^2 - 2x_V + 3 \\
2x_V^2 - 8x_V + 6 &= 0 && \mid : 2 \\
x_V^2 - 4x_V + 3 &= 0 && \mid p\text{-}q\text{-formula} \\
x_{C1/2} &= -\frac{-4}{2} \pm \sqrt{\left(\frac{-4}{2}\right)^2 - 3} \\
&= 2 \pm \sqrt{2^2 - 3} \\
&= 2 \pm 1 \\
x_{C1} = 2 - 1 = 1 \qquad & x_{C2} = 2 + 1 = 3
\end{aligned}$$

We get the corresponding y-values by putting the x-values into any desired functional equation, f_1 or f_2. My arbitrary choice is f_1.

$$y_{C1} = f_1(x_{C1}) = x_{C1}^2 - 6x_{C1} + 3 = 1^2 - 6 \cdot 1 + 3 = -2$$

$$y_{C2} = f_1(x_{C2}) = x_{C2}^2 - 6x_{C2} + 3 = 3^2 - 6 \cdot 3 + 3 = -6$$

We have got these both crossing points: $C_1(1|-2)$ and $C_2(3|-6)$

[4] See also here: http://www.dk4ek.de/lib/exe/fetch.php/lin.pdf

3 Examples

3.1 Example 1

A quadratic function has the vertex $V(-1|12)$ and a zero at $x_0 = -3$. What is the functional equation?

Solution: As we know the vertex, we should use the vertex form as a method of resolution.

$$\begin{aligned} f(x) &= a \cdot (x - x_V)^2 + y_V \\ &= a \cdot \big(x - (-1)\big)^2 + 12 \\ f(x) &= a \cdot (x + 1)^2 + 12 \end{aligned}$$

Now only the parameter $\boldsymbol{a}$ is missing. For the calculation we can use the known zero $x_0 = -3$. If we calculate the value of the function at $x_0 = -3$, we will get the value 0.

$$\begin{aligned} f(x_0) &= 0 \\ a \cdot (x_0 + 1)^2 + 12 &= 0 \\ a \cdot (-3 + 1)^2 + 12 &= 0 \\ a \cdot (-2)^2 + 12 &= 0 \\ 4a + 12 &= 0 \qquad | -12 \\ 4a &= -12 \quad | : 4 \\ a &= -3 \end{aligned}$$

Using this value we can show the funtional equation in vertex form.

$$f(x) = -3 \cdot (x + 1)^2 + 12$$

Now we have to transform the vertex form into the normal form. For that we transform the term in the brackets by using the first binomial formula.[5]

$$\begin{aligned} f(x) &= -3 \cdot (x + 1)^2 + 12 \\ &= -3 \cdot (x^2 + 2x + 1) + 12 \\ &= -3x^2 - 6x - 3 + 12 \\ f(x) &= -3x^2 - 6x + 9 \end{aligned}$$

We get the searched funktion: $\boxed{f(x) = -3x^2 - 6x + 9}$

[5] For details of the first binomial formula look here:
http://www.dk4ek.de/lib/exe/fetch.php/binom.pdf

3.2 Example 2

A parabula passes through the points $P_1(3|9)$ and $P_2(4|19)$. She hits the y-axis at $y_0 = 3$. Where do we find her vertex?

Solution: A parabula is a quadratic function. So we can start with the normal form of a quadratic function:

$$f(x) = ax^2 + bx + c$$

As the y-axis intercept is known as $y_0 = 3$, we also know the parameter c.

$$c = y_0 = 3$$

With this information we can concrete our functional equation:

$$f(x) = ax^2 + bx + 3$$

Now we can put the coordinates of the known points into the functional equation:

$$\begin{array}{lllrcl} P_1(3|9): & f(3) = 9 & \Rightarrow & a \cdot 3^2 + b \cdot 3 + 3 & = & 9 \\ P_2(4|19): & f(4) = 19 & \Rightarrow & a \cdot 4^2 + b \cdot 4 + 3 & = & 19 \end{array}$$

At first we simplify the two equations:

$$\begin{array}{lrcll} (1) & a \cdot 3^2 + b \cdot 3 + 3 & = & 9 & \\ (2) & a \cdot 4^2 + b \cdot 4 + 3 & = & 19 & \\ \hline (1) & 9a + 3b + 3 & = & 9 & | - 3 \\ (2) & 16a + 4b + 3 & = & 19 & | - 3 \\ \hline (1) & 9a + 3b & = & 6 & \\ (2) & 16a + 4b & = & 16 & \end{array}$$

This system of equations can be solved with any desired method.[6] Arbitrary I choose the method of insertion. I dissolve equation (1) to b and insert the result into equation (2).

$$\begin{array}{rcll} 9a + 3b & = & 6 & | - 9a \\ 3b & = & 6 - 9a & | : 3 \\ b & = & 2 - 3a & \end{array}$$

Insertion in (2):

$$\begin{array}{rcll} 16a + 4b & = & 16 & \\ 16a + 4 \cdot (2 - 3a) & = & 16 & \\ 16a + 8 - 12a & = & 16 & | - 8 \\ 4a & = & 8 & | : 4 \\ a & = & 2 & \end{array}$$

[6] See for details of methods to solve a system of equations here: http://www.dk4ek.de/lib/exe/fetch.php/lingl.pdf

We insert this result into the converted equation (1) to get b.

$$b = 2 - 3a = 2 - 3 \cdot 2 = 2 - 6 = -4$$

With theese results we can concrete the functional equation:

$$f(x) = 2x^2 - 4x + 3$$

Now we have to calculate the coordinates of the vertex. The vertex formula will help us, to get them.

$$\begin{aligned} x_V &= -\frac{b}{2a} \\ &= -\frac{-4}{2 \cdot 2} \\ x_V &= 1 \end{aligned}$$

The corresponding y-value y_V we get from the functional equation.

$$\begin{aligned} y_V &= f(x_V) \\ &= 2x_V^2 - 4x_V + 3 \\ &= 2 \cdot 1^2 - 4 \cdot 1 + 3 \\ &= 2 - 4 + 3 \\ y_V &= 1 \end{aligned}$$

The vertex we looked for is: $\boxed{V(1|1)}$

3.3 Example 3

A parabula with the shape factor $a = -1$ runs through the points $P_1(1|4)$ and $P_2(4|1)$. Calculate the vertex!

Solution: The normal form of the functional equation is:

$$f(x) = ax^2 + bx + c$$

As the shape factor is nown as $a = -1$, we can concrete the functional equation:

$$f(x) = -x^2 + bx + c$$

The coordinates of both points can be inserted.

$$\begin{array}{lcccl} P_1(1|4): & f(1) = 4 & \Rightarrow & -1^2 + b \cdot 1 + c & = & 4 \\ P_2(4|1): & f(4) = 1 & \Rightarrow & -4^2 + b \cdot 4 + c & = & 1 \end{array}$$

At first we simplify the two equations:

$$\begin{array}{lrcll} (1) & -1^2 + b \cdot 1 + c & = & 4 & \\ (2) & -4^2 + b \cdot 4 + c & = & 1 & \\ \hline (1) & -1 + b + c & = & 4 & | + 1 \\ (2) & -16 + 4b + c & = & 1 & | + 16 \\ \hline (1) & b + c & = & 5 & \\ (2) & 4b + c & = & 17 & \end{array}$$

This system of equations can be solved with any desired method.[7] As both equations have the same coefficient of c (it is 1), the method of subtraction should be beneficial, c immediately will disappear.

$$\begin{array}{lrrcll} (1) & b & +c & = & 5 & |- \\ (2) & 4b & +c & = & 17 & | \\ \hline & 3b & & = & 12 & | : 3 \\ & b & & = & 4 & \end{array}$$

To get the value of c we put this solution into equation (1) or (2). Because equation (1) seems simpler to me, I use this one.

$$\begin{array}{rcll} b + c & = & 5 & \\ 4 + c & = & 5 & | - 4 \\ c & = & 1 & \end{array}$$

With theese solutions we can present the functional equation:

$$f(x) = -x^2 + 4x + 1$$

[7] See for details of methods to solve a system of equations here: http://www.dk4ek.de/lib/exe/fetch.php/lingl.pdf

Now we have to calculate the coordinates of the vertex. The vertex formula will help us, to get them.

$$
\begin{aligned}
x_V &= -\frac{b}{2a} \\
&= -\frac{4}{2 \cdot (-1)} \\
x_V &= 2
\end{aligned}
$$

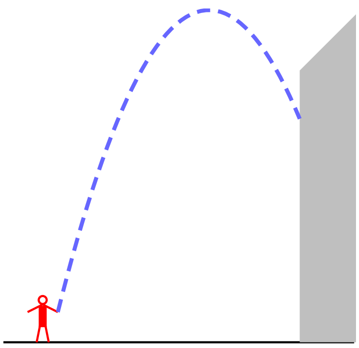
The corresponding y-value y_V we get from the functional equation.

$$
\begin{aligned}
y_V &= f(x_V) \\
&= -x_V^2 + 4x_V + 1 \\
&= -2^2 + 4 \cdot 2 + 1 \\
&= -4 + 8 + 1 \\
y_V &= 5
\end{aligned}
$$

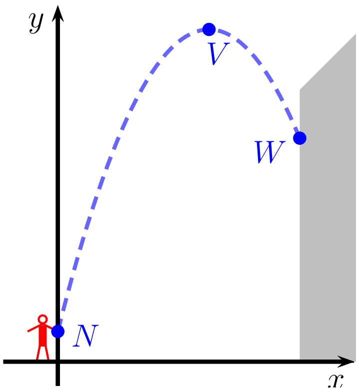
The vertex we looked for is: $\boxed{V(2|5)}$

3.4 Example 4

A firefighter wants to extinguish a fire in a building with the water jet comming out of his hose. The building has a distance of 8 meter to him. The water jet hits a window, 7,40 meter above the ground. The firefighter has the nozzle of the hose one meter above the ground in his hands. In a horizontal ditance of 5 meter the water jet has its maximum height. What is the value of the maximum height? The water jet has the shape of a parabula.



Solution: To solve this problem it is useful to place a coordinate system into the sketch plan. There are of course several positions to place it. But when we have decided a position, the functional equation will depend on this coordinate system.



In my example of solution I place the x-axis to the bottom of the earth. The y-axis I place through the nozzle of the hose in the firefighters hand. The point of the nozzle I mark with the letter N. The point, where the water jet hits the window, I mark with the letter W. V is the vertex.

For all furter calculations I use the unit *meter* for all dimensions. So it is possible to omit all dimensions during the calculations. Using the known data we can write down all points in this way:

$$N(0|1) \qquad W(8|7{,}4) \qquad V(5|y_V)$$

We have to find out the value of parameter y_V.

As the water jet has the shape of a **parabula**, the corrseponding function is a quadratic function. Therefore we get thwo different concepts for the solution:

1. Vertex form
2. Normal form

Variant 1 of solution: Let us start with the **vertex form** of the quadratic function.

$$f(x) = a \cdot (x - x_V)^2 + y_V$$

The already known value $x_V = 5$ will be insertet.

$$f(x) = a \cdot (x - 5)^2 + y_V$$

Now we can insert the parameters of the known points $L(0|1)$ and $F(8|7{,}4)$ into the functional equation.

$$\begin{array}{rlcrcl} L(0|1): & f(0) = 1 & \Rightarrow & a \cdot (0-5)^2 + y_V & = & 1 \\ F(8|7{,}4): & f(8) = 7{,}4 & \Rightarrow & a \cdot (8-5)^2 + y_V & = & 7{,}4 \end{array}$$

Both equations – I name them equation (1) and (2) – will now be simplified.

$$\begin{array}{crcl} (1) & a \cdot (0-5)^2 + y_V & = & 1 \\ (2) & a \cdot (8-5)^2 + y_V & = & 7{,}4 \\ \hline (1) & a \cdot 5^2 + y_V & = & 1 \\ (2) & a \cdot 3^2 + y_V & = & 7{,}4 \\ \hline (1) & 25a + y_V & = & 1 \\ (2) & 9a + y_V & = & 7{,}4 \end{array}$$

This system of equations can be solved with any desired method. As both equations have the same coefficient of c (it is 1), the method of subtraction[8] should be beneficial. y_V immediately will disappear, if equation (2) is subtracted from equation (1), as in both equations its coefficient is equal.

$$\begin{array}{ccccll} (1) & 25a & +y_V & = & 1 & | \\ (2) & 9a & +y_V & = & 7{,}4 & |- \\ \hline & 16a & & = & -6{,}4 & |:16 \\ & a & & = & -0{,}4 & \end{array}$$

To get the value of y_V we insert this value $a = -0{,}4$ into equation (1) or (2). I have choosen equation (1).

$$\begin{array}{rcll} 25a + y_V & = & 1 & \\ 25 \cdot (-0{,}4) + y_V & = & 1 & \\ -10 + y_V & = & 1 & |+10 \\ y_V & = & 11 & \end{array}$$

[8]See for details of method of subtraction here:
http://www.dk4ek.de/lib/exe/fetch.php/add.pdf

Variant 2 of solution: Now we use the **normal form** to solve the problem.

$$f(x) = ax^2 + bx + c$$

In this variant of solution we have do get not only two parameters (a and y_V), we need **tree** of them (a, b and c) to get the functional equation.

At first we can – similar to variant of solution 1 – insert the coordinates of the points $L(0|1)$ and $F(8|7{,}4)$ into the normal form.

$$\begin{aligned} L(0|1): \quad f(0) = 1 \quad &\Rightarrow \quad a \cdot 0^2 + b \cdot 0 + c = 1 \\ F(8|7{,}4): \quad f(8) = 7{,}4 \quad &\Rightarrow \quad a \cdot 8^2 + b \cdot 8 + c = 7{,}4 \end{aligned}$$

Now we simplify both equations.

$$\begin{array}{llcll} (1) & 0a + 0b + c & = & 1 & \Rightarrow \quad c = 1 \\ (2) & 64a + 8b + c & = & 7{,}4 & \end{array}$$

We got immediately parameter $c = 1$ out of equation (1). This value we can insert into equation (2). Afterwards we simplify the equation.

$$\begin{array}{llcll} (2) & 64a + 8b + c & = & 7{,}4 & \\ (2) & 64a + 8b + 1 & = & 7{,}4 & | - 1 \\ (2) & 64a + 8b & = & 6{,}4 & \end{array}$$

We did not use the fact, that we know the x-value of the vertex with $x_V = 5$. This value can be inserted into the vertex formula.

$$\begin{aligned} x_V &= -\frac{b}{2a} \\ 5 &= -\frac{b}{2a} \end{aligned}$$

It is simple to rearrange this equation to b. We insert the result into the tranformed equation (2). This work is called **method of insertion**.[9]

$$\begin{aligned} 5 &= -\frac{b}{2a} \quad | \cdot (-2a) \\ -10a &= b \end{aligned}$$

Insert into equation (2):

$$\begin{aligned} 64a + 8b &= 6{,}4 \\ 64a + 8 \cdot (-10a) &= 6{,}4 \\ 64a - 80a &= 6{,}4 \\ -16a &= 6{,}4 \quad | : (-16) \\ a &= -0{,}4 \end{aligned}$$

[9]For details of the method of insertion look here:
http://www.dk4ek.de/lib/exe/fetch.php/einsetz.pdf

We insert this result into the tranformed vertex formula.

$$\begin{aligned} b &= -10a \\ &= -10 \cdot (-0{,}4) \\ b &= 4 \end{aligned}$$

With this result we can write down the functional equation.

$$f(x) = -0{,}4x^2 + 4x + 1$$

We insert the known value $x_V = 5$ as x in the functional equation to get the y-value y_V of the vertex.

$$\begin{aligned} y_V &= f(x_V) \\ &= -0{,}4x_V^2 + 4x_V + 1 \\ &= -0{,}4 \cdot 5^2 + 4 \cdot 5 + 1 \\ &= -10 + 20 + 1 \\ y_V &= 11 \end{aligned}$$

Comparing both methods of solution we probably notice, that the first variant of solution is a little bit shorter. In any case we get the result:

The water jet reaches a maximum height of 11 meter.

3.5 Example 5

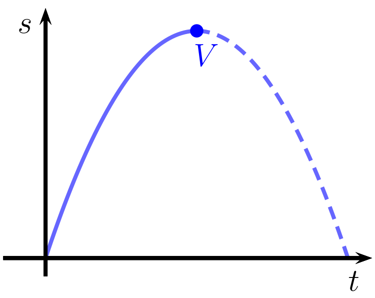
A child is suddenly running in the street. A car driver, that drives with the allowed speed in built-up aereas of 50 $\frac{\text{km}}{\text{h}}$, just in time can stop in front of the child. What would bee the speed by hitting the child, if he is "a little bit" too fast with a speed of 70 $\frac{\text{km}}{\text{h}}$?

Solution: At first we need some basical principles. The way s, that a car moves in the time t while braking, is a quadratic function. Here it is:

$$s(t) = -\frac{1}{2} \cdot a \cdot t^2 + v_0 \cdot t$$

<u>Attention!</u> Parameter a in this formula is not identical with parameter a in the normal form of the quadratic function. In this context a is the value of the brake retardation and v_0 is the start value of the speed of the car. The speed lowers as a linear function:

$$v(t) = v_0 - a \cdot t$$



On the right you see the course of the way-time-function of the braking process. In the coordinate origin at the time $t = 0$ the braking process is starting. The vehicle at first is movong forwards. The steep curve progression at the beginning on the left implifys a high speed. The curve progression changes more and more to flat, the speed decreases. At the vertex marked with C the vehicle stops. As the brake process is ending at this point, the definition aerea of the quadratic function ends here. If a force like the braking force would pull furthermore the car backwards, the car would be ascelerated backwards, as the dashed part of the function graph shows. As this does not happen, the graph ist plotted dashed.

Under normal conditions we can calculate with a brake retardation of appproximate 4 $\frac{\text{m}}{\text{s}^2}$.

To get a more comfortable calculation I want to omit the units. As all times should be given in seconds and all distances in meters, we have to use the unit "meters per second" for all speeds. It in not necessary to transform the units, when we remember: $1\,\frac{\text{m}}{\text{s}} = 3{,}6\,\frac{\text{km}}{\text{h}}$ We only have to place the factor 3,6 into our formulas. We will get the following forms of the formulas. We begin with the way-time-function:

$$s(t) = -\frac{1}{2} \cdot a \cdot t^2 + \frac{v_0}{3{,}6} \cdot t$$

And tis is the speed-time-function:

$$\begin{aligned} \frac{v(t)}{3{,}6} &= \frac{v_0}{3{,}6} - a \cdot t \qquad | \cdot 3{,}6 \\ v(t) &= v_0 - 3{,}6 \cdot a \cdot t \end{aligned}$$

We remember: When we use this form of the formulas, we put s in meters, t in seconds, a in $\frac{\mathrm{m}}{\mathrm{s}^2}$ and v in $\frac{\mathrm{km}}{\mathrm{h}}$ into the formulas.

So we get this method of solution:

1. We calculate the time until the car stops for the maximum allowed speed by calculating the t_S-value (x_S-value) of the vertex with the use of the vertex formula.

2. By using this time as t-value (x-value) we can calculate the braking distance s_S (y_S) with the way-time-function of the maximum allowed speed.

3. Now we take this braking distance s_S and the way-time-function of the fast speed to calculate the time t_c up to the crash.

4. By the use of this time in the speed-time-function we can calculate the speed, with that the car will hit the child.

At first we put the suggested value of the brake retardation with $4\,\frac{\mathrm{m}}{\mathrm{s}^2}$ into the formulas and simplify them.

$$\begin{aligned} s(t) &= -\frac{1}{2}\cdot a\cdot t^2+\frac{v_0}{3{,}6}\cdot t \\ s(t) &= -\frac{1}{2}\cdot 4\cdot t^2+\frac{v_0}{3{,}6}\cdot t \\ s(t) &= -2\cdot t^2+\frac{v_0}{3{,}6}\cdot t \end{aligned}$$

$$\begin{aligned} v(t) &= v_0-3{,}6\cdot a\cdot t \\ v(t) &= v_0-3{,}6\cdot 4\cdot t \\ v(t) &= v_0-14{,}4\cdot t \end{aligned}$$

Now we calculate the brake time with the allowed speed. This time is the t-value t_S of the vertex.

$$\begin{aligned} t_S &= -\frac{b}{2a} \\ &= -\frac{\frac{50}{3{,}6}}{2\cdot(-2)} \\ t_S &= 3{,}472 \end{aligned}$$

After a time of 3,472 seconds the car stops. We calculate the braking distance s_S.

$$\begin{aligned} s_S &= -2\cdot t_S^2+\frac{v_0}{3{,}6}\cdot t_S \\ &= -2\cdot(3{,}472)^2+\frac{50}{3{,}6}\cdot 3{,}472 \\ s_S &= 24{,}11 \end{aligned}$$

When driving with the allowed speed of $50\,\frac{\mathrm{km}}{\mathrm{h}}$ the braking distance is 24,11 m.

Now we calculate the time while breaking at the initial speed of $70\,\frac{\mathrm{km}}{\mathrm{h}}$, until the car covers the distance of 24,11 m. The quadratic equation, that occures during solution, we can solve with the p-q-formula[10].

$$
\begin{aligned}
s &= -2 \cdot t^2 + \tfrac{v_0}{3{,}6} \cdot t \\
24{,}11 &= -2 \cdot t^2 + \tfrac{70}{3{,}6} \cdot t && \mid + 2 \cdot t^2 - \tfrac{70}{3{,}6} \\
2 \cdot t^2 - \tfrac{70}{3{,}6} \cdot t + 24{,}11 &= 0 && \mid : 2 \\
t^2 - 9{,}72 \cdot t + 12{,}055 &= 0 \\
t_{1/2} &= -\tfrac{-9{,}72}{2} \pm \sqrt{\left(\tfrac{-9{,}72}{2}\right)^2 - 12{,}055} \\
t_{1/2} &= 4{,}86 \pm 3{,}40 \\
t_1 = 1{,}46 & \quad t_2 = 8{,}26
\end{aligned}
$$

Now probably the question appears, why we have got **two** solutions. The reason is as following:

As in the previous text already described, the meaning of "braking" is a force, that pulls on the car backwards. Hereby the car slows down, until it stops. If now still a force would work pulling from behind, the car would be ascelerated backwards and would cross some time later the milestone at 24,11 m. Therefore the formula looses its validity at the point of stopping, all greater times are **beyond** the validity of the formula. Therefore we have to calculate with the smaller time $t_1 = 1{,}46\,\mathrm{s}$.

$$
\begin{aligned}
v &= v_0 - 14{,}4 \cdot t \\
&= 70 - 14{,}4 \cdot 1{,}46 \\
v &= 48{,}98
\end{aligned}
$$

The child would be hit with a speed of $48{,}98\,\frac{\mathrm{km}}{\mathrm{h}}$. This is rounabout the allowed speed. So a "little bit" faster has very severe consequences for the child.

Appendix: What would have happened, if we had calculated with the greater time $t_2 = 8{,}26\,\mathrm{s}$? We just try it out.

$$
\begin{aligned}
v &= v_0 - 14{,}4 \cdot t \\
&= 70 - 14{,}4 \cdot 8{,}26 \\
v &= -48{,}94
\end{aligned}
$$

The result is (apart from rounding errors) almost the same as calculating with the smaller time, only the algebraic sign is negative. With this speed the car would cross **backwards** the milestone at 24,11 m, if the backwards directed braking force would still work after the stop. But we know, it does **not** work afterwards.

[10] For details of the p-q-formula look here: http://www.dk4ek.de/lib/exe/fetch.php/quad.pdf

4 Exercises

4.1 Exercise 1:

Specify a quadratic function $f(x)$ with the shape factor 1 and a vertex $S(4|-7)$!

4.2 Exercise 2:

You have got the function $f_1(x) = 2x^2 - 4x + 3$. Specify the function $f_2(x)$, that is shifted comparing to $f_1(x)$ 3 units to the right and 2 units downwards!

4.3 Exercise 3:

The quadratic function $f(x)$ has the vertex $V(4|1)$. The functional graph hits the y-axis at $y_0 = -7$. Calculate the functional equation $f(x)$.

4.4 Exercise 4:

Calculate the vertex, the zeros and the range of values of the quadratic function:

$$f(x) = 3x^2 - 12x + 15$$

4.5 Exercise 5:

Find out the vertex, the zeros and the range of values of the quadratic function:

$$f(x) = -16x^2 - 16x + 5$$

4.6 Exercise 6:

Find out the crossing points of the parabula of the function $f_1(x) = 4x^2 - 9x + 1$ with the straight line of the function $f_2(x) = 3x + 17$!

4.7 Exercise 7:

Find out the crossing points of the parabula of the function $f_1(x) = 9x^2 + 12x - 4$ with the straight line of the function $f_2(x) = -12x + 5$!

4.8 Exercise 8:

You have the quadratic function $f(x) = -2x^2 + 5x - 3$. Specify the vertex and find out inverse function $f^{-1}(x)$. What is the the definition range of this inverse function?

4.9 Exercise 9:

Find out the crossing points between the parabulas with the functional equations $f_1(x) = 4x^2 + 3x - 8$ and $f_2(x) = 7x^2 + 9x + 7$.

4.10 Exercise 10:

Find out the functional equation of the parabula, that crosses these points: $P_1(-1|8)$, $P_2(2|-1)$ und $P_3(4|3)$

4.11 Exercise 11:

Find out the functional equation of the linear function $f_1(x)$, whose straight line touches the parabula with the functional equation $f_1(x) = x^2 - 4x + 4$ at $x_b = 4$ as a tangent.

4.12 Exercise 12:

A parabula has the vertex $V(3|2)$ and runs through the point $P(5|10)$. Find out its funktional equation $f(x)$!

4.13 Exercise 13:

Calculate the zeros and the vertex of the quadratic function (as far they exist):

$$f(x) = -16x^2 + 24x - 25$$

4.14 Exercise 14:

The quadratic function $f(x)$ has the vertex $S(4|3)$ and runs through the point $P(6|11)$. Find out the functional equation of $f(c)$!

4.15 Exercise 15:

Find out the functional equation of the parabula, that crosses these points: $P_1(-3|32)$, $P_2(-1|10)$ and $P_3(2|7)$

4.16 Exercise 16:

A shifted normal parabula (factor of shape $a = 1$) runs through the points $P_1(-1|-2)$ und $P_2(0|1)$. What is the associated functional equation $f(x)$?

4.17 Exercise 17:

Find out the functional equation of the parabula, that crosses these points: $P_1(0|8)$, $P_2(1|3)$ and $P_3(2|0)$

4.18 Exercise 18:

A parabula (quadratic function) has the vertex $S(-2|0)$ and runs through the point $P(-1|-2)$. Specify the functional equation $f(x)$!

4.19 Exercise 19:

Take the parabula of the function $f_1(x) = \frac{1}{2}(x-4)^2 - 1$ and shift her so far **upwards**, that she will hit the point $P(2|3)$. What ist the functional equation $f_2(x)$ of the new function?

4.20 Exercise 20:

You should shift the parabula of the quadratic equation $f_1(x) = \frac{1}{2}x^2 - 4x + 7$ so far to the **left**, that the new parabula hits the point $P(0|\frac{7}{2})$. Specify the corresponding functional equation $f_2(x)$. Show **all** possible solutions!

4.21 Exercise 21:

How has the parabula with the functional equation $f_1(x) = \frac{1}{2}x^2 - 4x + 7$ to be changed, that the point $P(3|-6)$ belongs to the new parabula? The new parabula should have the **same vertex** als the old one. Secify the functional equation $f_2(x)$ of the new parabula!

4.22 Exercise 22:

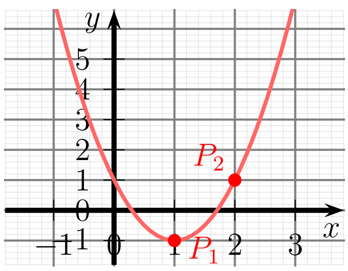
The parabula of the quadratic function $f_1(x)$ crosses the straight line of the function $f_2(x) = x - 2$ at $x_1 = -4$ and $x_2 = 1$. The parabula crosses the y-axis at $y_0 = 2$. Specify the functional equation $f_1(x)$!

4.23 Exercise 23:

Calculate the crossing points – if they exist – of the two parabulas with the functional equations $f_1(x) = x^2 - 2x + 3$ and $f_2(x) = 2x^2 - 8x + 12$.

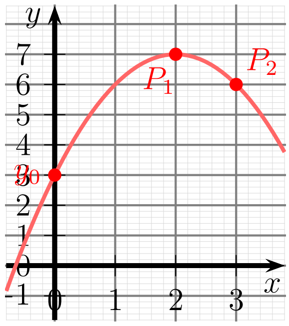
4.24 Exercise 24:

A parabula with the shape factor $a = 2$ runs through the points $P_1(1|-1)$ and $P_2(2|1)$. Specify the functional equation $f(x)$!



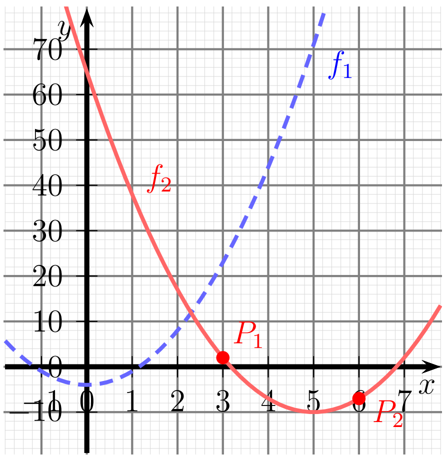
4.25 Exercise 25:

A parabula crosses the points $P_1(2|7)$ und $P_2(3|6)$ and cuts the y-axis at $y_0 = 3$. Find out the functional equation $f(x)$!



4.26 Exercise 26:

Shift the parabula with the functional equation $f_1(x) = 3x^2 - 4$ in a way, that the new parabula crosses the points $P_1(3|2)$ und $P_2(6|-7)$. Specify the functional equation $f_2(x)$, that occured by this shifting.



4.27 Exercise 27:

A quadratic function has the shape factor $a = -1$ and the vertex $V(2|3)$. Specify the functional equation $f(x)$ in **vertex form and in normal form**!

4.28 Exercise 28:

The functional graph of a quadratic function has the vertex $V(3|-1)$ and runs through the point $P(1|7)$. Specify the functional equation $f(x)$ in **vertex form and in normal form**!

4.29 Exercise 29:

We have a quadratic function with the functional equation $f(x) = -2x^2 - 4x + 6$. Calculate the vertex and the zeros!

4.30 Exercise 30:

A parabula with the vertex $V(4|7)$ crosses the y-axis at $y_0 = 4$. Specify the associated functional equation!

4.31 Exercise 31:

We have a quadratic function with the functional equation:

$$f_1(x) = 3x^2 + 24x + 53$$

We are looking for the function f_2 with the same vertex as f_1. The related parabula of f_2 should cross the point $P(-2|-3)$.

4.32 Exercise 32:

A parabula crosses the points $P_1(0|14)$ and $P_2(2|-2)$. The vertex can be found at $x_V = 3$. Specify the related functional equation $f(x)$!

4.33 Exercise 33:

Which of these five functions have the same vertex?

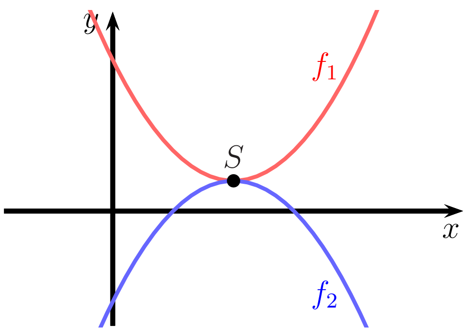
$$\begin{aligned} f_1(x) &= x^2 - 10x + 21 \\ f_2(x) &= 3x^2 - 30x + 71 \\ f_3(x) &= -2x^2 + 20x - 48 \\ f_4(x) &= 0{,}5x^2 + 5x + 8{,}5 \\ f_5(x) &= 4 \cdot (x-5)^2 - 4 \end{aligned}$$

4.34 Exercise 34:

We have given the function f_1 with the functional equation

$$f_1(x) = 4x^2 - 24x + 37$$

We want to find out the functional equation f_2 of a parabula, that is produced by turning the parabula of f_1 with an angle of 180° around the vertex, as shown in the graphic on the right side.



4.35 Exercise 35:

Find out the crossing points of the parabula with $f_1(x) = 10x^2 + 12x - 4$ and the stright line with $f_2(x) = 9x - 3$.

5 Solutions

5.1 Exercise 1:

Specify a quadratic function $f(x)$ with the shape factor 1 and a vertex $S(4|-7)$!

Solution: It is appropriate to use the vertex formula.

$$f(x) = a \cdot (x - x_V)^2 + y_V$$

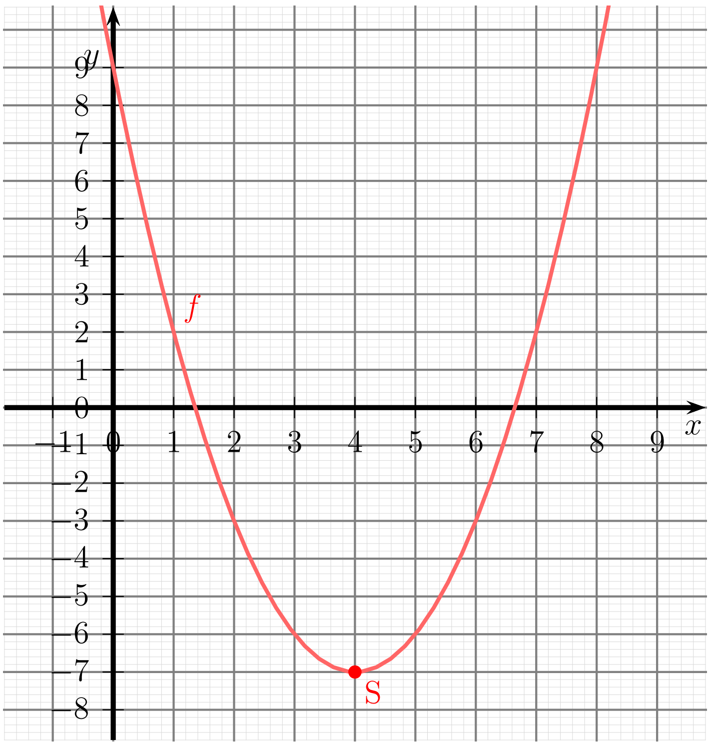
Now we can insert the known coordinates of the vertex and the also known shape factor, and we are already completed.

$$f(x) = 1 \cdot (x - 4)^2 - 7$$

If you want, you can dissolve the brackets, to transform the functional equation into the normal form.

$$\begin{aligned} f(x) &= 1 \cdot (x-4)^2 - 7 \\ &= x^2 - 8x + 16 - 7 \\ f(x) &= x^2 - 8x + 9 \end{aligned}$$

Subsequent you see the associated functional graph.



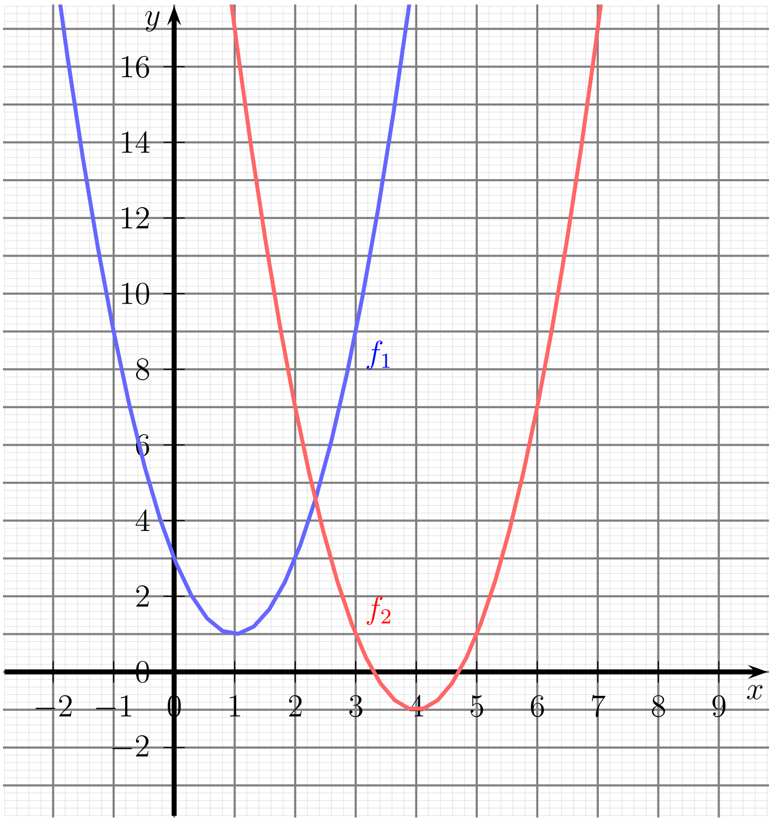
5.2 Exercise 2:

You have got the function $f_1(x) = 2x^2 - 4x + 3$. Specify the function $f_2(x)$, that is shifted comparing to $f_1(x)$ 3 units to the right and 2 units downwards!

Solution: To solve this problem it is appropriate to use the formula of shifted function. 2 units downwards correspond to -2 units upwards.

$$\begin{aligned}
f_2(x) &= f_1(x - x_s) + y_s \\
&= f_1(x - 3) + (-2) \\
&= 2 \cdot (x - 3)^2 - 4 \cdot (x - 3) + 3 - 2 \\
&= 2 \cdot (x^2 - 6x + 9) - 4x + 12 + 1 \\
&= 2x^2 - 12x + 18 - 4x + 13 \\
f_2(x) &= 2x^2 - 16x + 31
\end{aligned}$$

Subsequent you see the associated functional graph.



5.3 Exercise 3:

The quadratic function $f(x)$ has the vertex $V(4|1)$. The functional graph hits the y-axis at $y_0 = -7$. Calculate the functional equation $f(x)$.

Solution: As we know the vertex, it is appropriate, to use the vertex form. We only have to calculate then the shape factor a.

$$\begin{aligned} f(x) &= a \cdot (x - x_V)^2 + y_V \\ f(x) &= a \cdot (x-4)^2 + 1 \end{aligned}$$

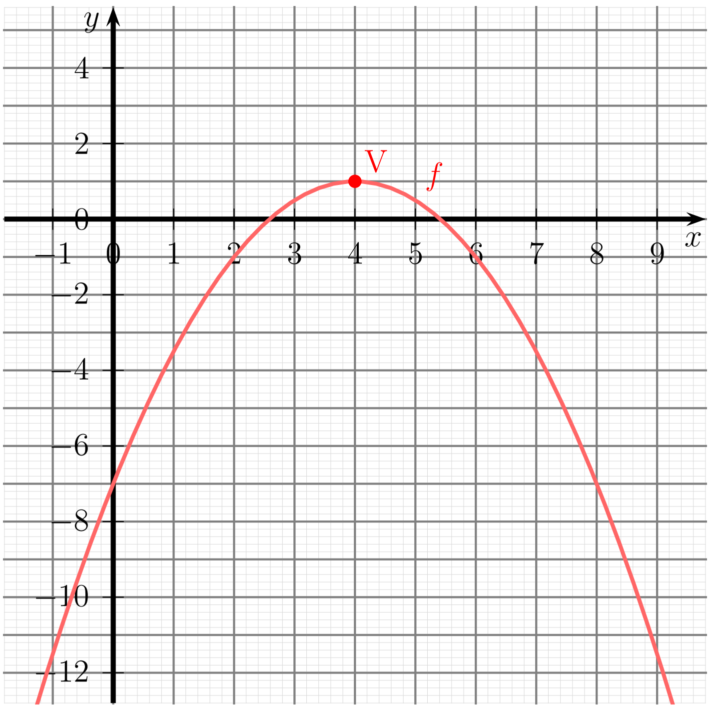
The crossing point with the y-axis means: $f(0) = y_S$. We insert this into the equation.

$$\begin{aligned} -7 &= a \cdot (0-4)^2 + 1 && \\ -7 &= a \cdot 16 + 1 && |-1 \\ -8 &= 16a && |:16 \\ a &= -\frac{8}{16} && \\ a &= -\frac{1}{2} && \end{aligned}$$

By using this value for a we get the wanted functional equation:

$$f(x) = -\frac{1}{2} \cdot (x-4)^2 + 1 = -\frac{1}{2} \cdot (x^2 - 8x + 16) + 1 = -\frac{1}{2}x^2 + 4x - 7$$

Here we see the course of the functional graph.



5.4 Exercise 4:

Calculate the vertex, the zeros and the range of values of the quadratic function:

$$f(x) = 3x^2 - 12x + 15$$

Solution: At a zero the value of the function is 0. Therefore the beginning of a calculation of zeros is: $f(x_0) = 0$.

$$\begin{aligned} 3x_0^2 - 12x_0 + 15 &= 0 \qquad \qquad |:3 \\ x_0^2 - 4x_0 + 5 &= 0 \\ x_{01/2} &= 2 \pm \sqrt{4-5} \\ x_{01/2} &= 2 \pm \sqrt{-1} \end{aligned}$$

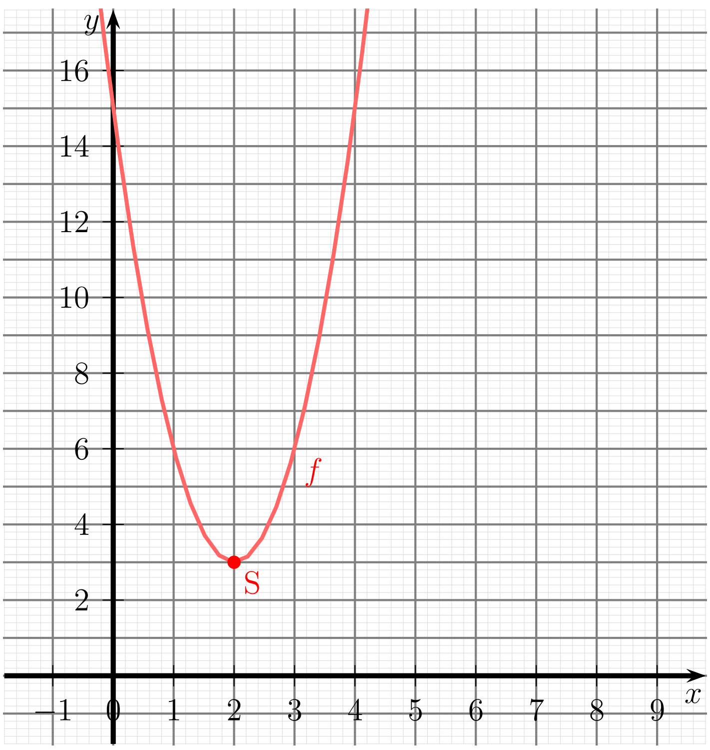
As there is no real solution for the sqare root, we have **no zeros**. On the other hand we can read the x-value of the vertex in this formula. It is always the value before $\pm$ and the root, here $x_V = 2$. We get the corresponding y-value y_V by inserting the value of x_V into the functional equation.

$$\begin{aligned} y_V &= f(x_V) \\ &= 3 \cdot 2^2 - 12 \cdot 2 + 15 \\ y_V &= 3 \qquad \qquad \Rightarrow \quad V(2|3) \end{aligned}$$

As the shape factor with the value $a = 3$ is positive, the parabula is opened upwards. Therefore we get the range of values:

$$W = \{y|y \geq 3\}$$

Here we see the course of the functional graph.



5.5 Exercise 5:

Find out the vertex, the zeros and the range of values of the quadratic function:

$$f(x) = -16x^2 - 16x + 5$$

Solution: At a zero the value of the function is 0. Therefore the beginning of a calculation of zeros is: $f(x_0) = 0$.

$$\begin{aligned}
-16x_0^2 - 16x_0 + 5 &= 0 \qquad\qquad | : (-16)\\
x_0^2 + x_0 - \frac{5}{16} &= 0\\
x_{01/2} &= -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{5}{16}}\\
x_{01/2} &= -\frac{1}{2} \pm \sqrt{\frac{4}{16} + \frac{5}{16}}\\
x_{01/2} &= -\frac{1}{2} \pm \sqrt{\frac{9}{16}}\\
x_{01/2} &= -\frac{1}{2} \pm \frac{3}{4}\\
x_{01/2} &= -\frac{2}{4} \pm \frac{3}{4}\\
x_{01} &= \frac{1}{4}\\
x_{02} &= -\frac{5}{4}
\end{aligned}$$

As we did it at exercise 4 we can read from this formula the x-value of the vertex as the value in front of the root. That is:

$$x_V = -\frac{1}{2}$$

We get the corresponding y-value y_V by inserting x_V into the functional equation:

$$y_V = f(x_V) = -16 \cdot \left(-\frac{1}{2}\right)^2 - 16 \cdot \left(-\frac{1}{2}\right) + 5 = 9$$

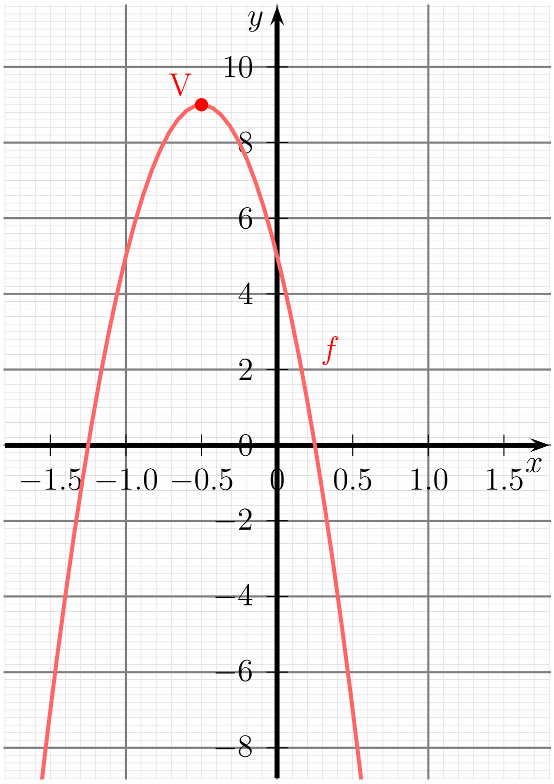
Hereby we got the vertex:

$$V\left(-\tfrac{1}{2}|9\right)$$

As the shape factor $a = -16$ is **negative**, the parabula is opened downwards. The range of values therefore is **below** the vertex:

$$W = \{x|x \leq 9\}$$

Here we see the course of the functional graph.



5.6 Exercise 6:

Find out the crossing points of the parabula of the function $f_1(x) = 4x^2 - 9x + 1$ with the straight line of the function $f_2(x) = 3x + 17$!

Solution: The crossing points are exact those points of both curves, where the x- and the y-values are identic. Therefore we can equate both functional terms to get them.

$$\begin{aligned}
f_1(x_S) &= f_2(x_S) \\
4x_S^2 - 9x_S + 1 &= 3x_S + 17 && |-3x_S - 17 \\
4x_S^2 - 12x_S - 16 &= 0 && |:4 \\
x_S^2 - 3x_S - 4 &= 0 \\
x_{S1/2} &= \frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{16}{4}} \\
&= \frac{3}{2} \pm \sqrt{\frac{25}{4}} \\
&= \frac{3}{2} \pm \frac{5}{2} \\
x_{S1} &= \frac{3}{2} + \frac{5}{2} = 4 \\
x_{S2} &= \frac{3}{2} - \frac{5}{2} = -1
\end{aligned}$$

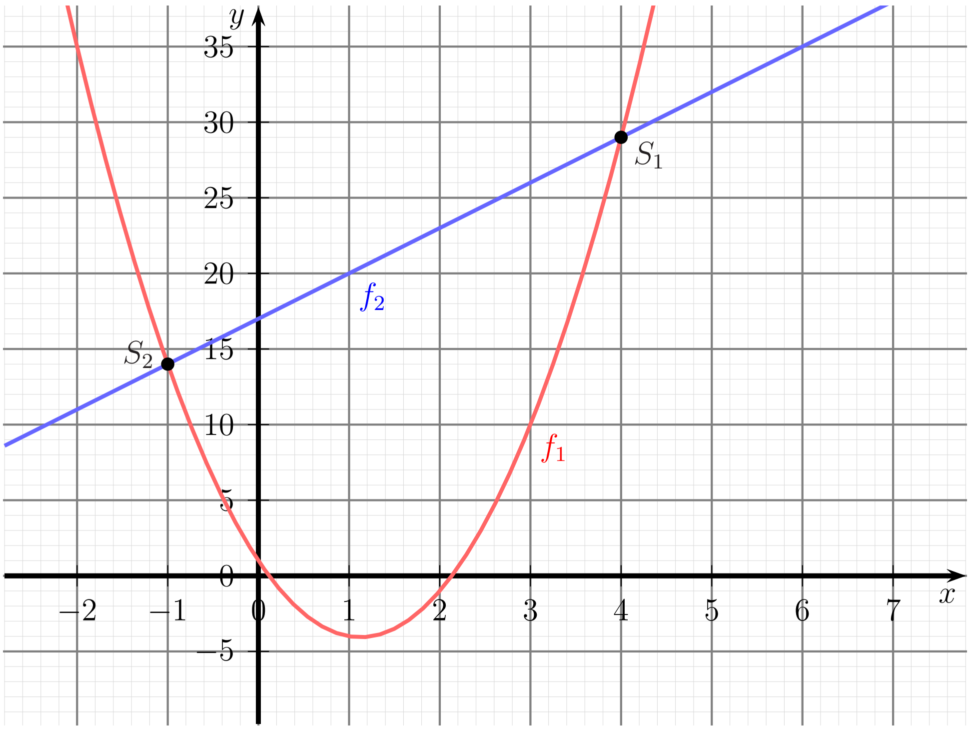
We get the corresponding y-values by inserting the x-values in any desired functional equation. I chose f_2 for that purpuse, it looks a bit simpler.

$$\begin{aligned}
y_S &= f_2(x_S) \\
y_{S1} &= 3x_{S1} + 17 = 3 \cdot 4 + 17 = 29 \\
y_{S2} &= 3x_{S2} + 17 = 3 \cdot (-1) + 17 = 14
\end{aligned}$$

We have got the crossing points:

$$\boxed{S_1(4|29)} \quad \text{and} \quad \boxed{S_2(-1|14)}$$

Here we see the courses of the functional graphs.



5.7 Exercise 7:

Find out the crossing points of the parabula of the function $f_1(x) = 9x^2 + 12x - 4$ with the straight line of the function $f_2(x) = -12x + 5$!

Solution: The crossing points are exact those points of both curves, where the x- and the y-values are identic. Therefore we can equate both functional terms to get them.

$$\begin{aligned}
f_1(x_S) &= f_2(x_S) \\
9x_S^2 + 12x_S - 4 &= -12x_S + 5 \qquad |+12x_S - 5 \\
9x_S^2 + 24x_S - 9 &= 0 \qquad |:9 \\
x_S^2 + \frac{8}{3}x_S - 1 &= 0 \\
x_{S1/2} &= -\frac{4}{3} \pm \sqrt{\frac{16}{9} + \frac{9}{9}} \\
&= -\frac{4}{3} \pm \sqrt{\frac{25}{9}} \\
&= -\frac{4}{3} \pm \frac{5}{3} \\
x_{S1} &= -\frac{4}{3} - \frac{5}{3} = -3 \\
x_{S2} &= -\frac{4}{3} + \frac{5}{3} = \frac{1}{3}
\end{aligned}$$

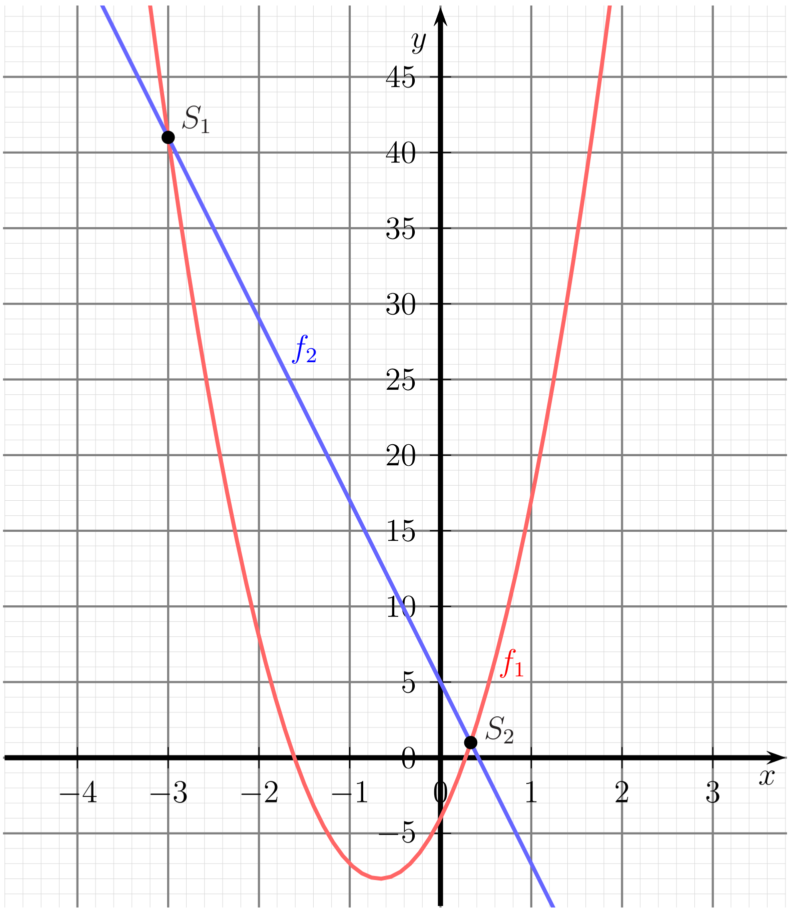
We get the corresponding y-values by inserting the x-values in any desired functional equation. I chose f_2 for that purpuse, it looks a bit simpler.

$$\begin{aligned}
y_S &= f_2(x_S) \\
y_{S1} &= -12x_{S1} + 5 = -12 \cdot (-3) + 5 = 41 \\
y_{S2} &= -12x_{S2} + 5 = -12 \cdot \frac{1}{3} + 5 = 1
\end{aligned}$$

We have got the crossing points:

$$\boxed{S_1(-3|41)} \quad \text{and} \quad \boxed{S_2\left(\tfrac{1}{3}\middle|1\right)}$$

Here we see the courses of the functional graphs.



5.8 Exercise 8:

You have the quadratic function $f(x) = -2x^2 + 5x - 3$. Specify the vertex and find out inverse function $f^{-1}(x)$. What is the the definition range of this inverse function?

Solution: We get the vertex easy by using the vertex formula.

$$x_S = -\frac{b}{2a} = -\frac{5}{2 \cdot (-2)} = \frac{5}{4}$$

$$y_S = f(x_S) = -2x_S^2 + 5x_S - 3 = -2\left(\frac{5}{4}\right)^2 + 5 \cdot \frac{5}{4} - 3 = \frac{1}{8}$$

$$S\left(\tfrac{5}{4}\middle|\tfrac{1}{8}\right)$$

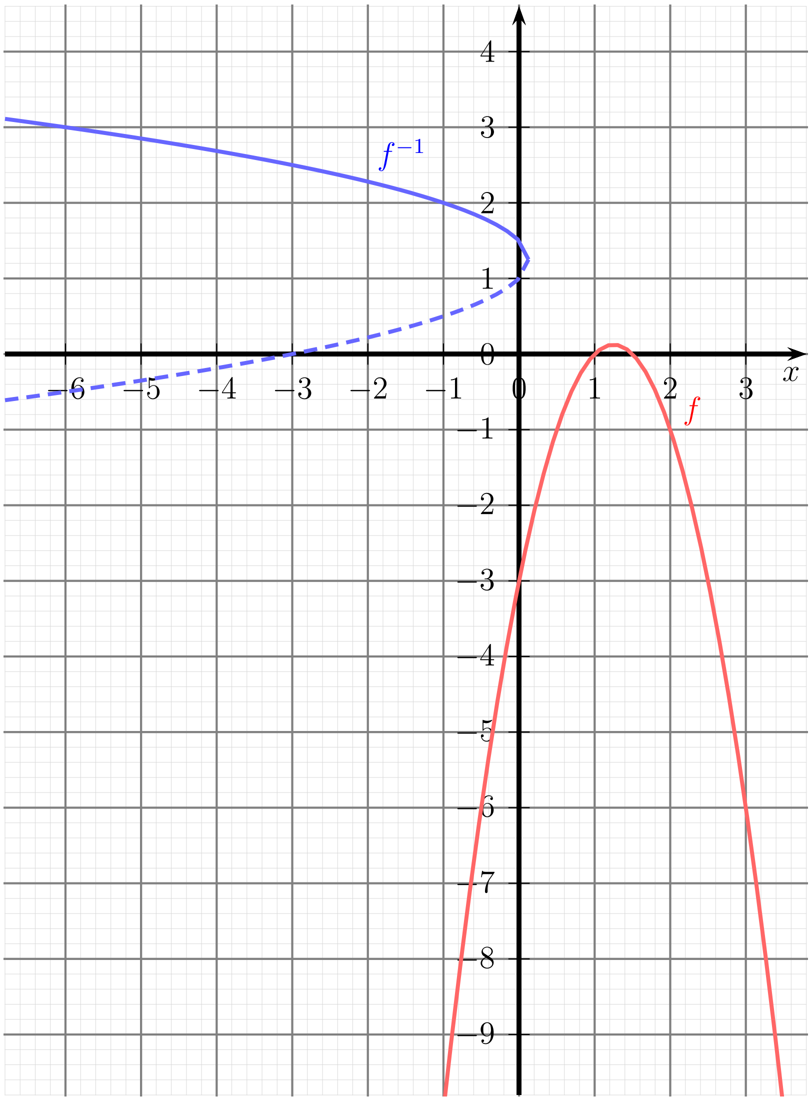
To get the inverse function $f^{-1}(x)$ we jut have to change the positions of x and y. Afterwards we dissolve the equation to y.

$$\begin{array}{rcll}
y &=& -2x^2 + 5x - 3 & \mid \text{change } x \text{ and } y \\
x &=& -2y^2 + 5y - 3 & \mid + 2y^2 - 5y + 3 \\
2y^2 - 5y + 3 + x &=& 0 & \mid : 2 \\
y^2 - \frac{5}{2}y + \frac{3}{2} + \frac{x}{2} &=& 0 & \mid p\text{-}q\text{-formula} \\
y_{1/2} &=& \frac{5}{4} \pm \sqrt{\frac{25}{16} - \frac{24}{16} - \frac{8x}{16}} & \\
&=& \frac{5}{4} \pm \sqrt{\frac{1-8x}{16}} & \\
&=& \frac{5}{4} \pm \frac{\sqrt{1-8x}}{4} & \\
f^{-1}(x) &=& \frac{5}{4} \pm \frac{1}{4}\sqrt{1-8x} &
\end{array}$$

$$\begin{array}{rcll}
1 - 8x &\geq& 0 & \mid -1 \\
-8x &\geq& -1 & \mid : (-8) \\
x &\leq& \frac{1}{8} &
\end{array}$$

$$\boxed{D = \left\{x \middle| x \leq \tfrac{1}{8}\right\}}$$

Here we see the courses of the functional graphs.



5.9 Exercise 9:

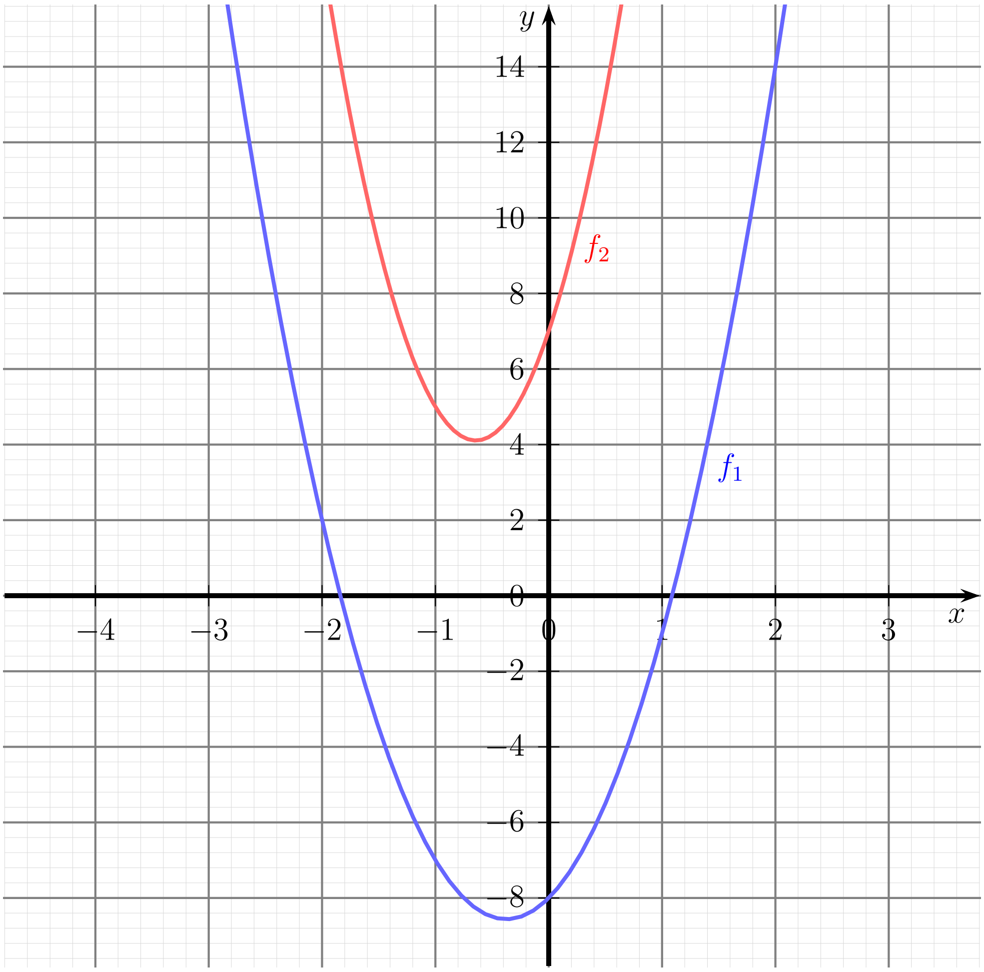
Find out the crossing points between the parabulas with the functional equations $f_1(x) = 4x^2 + 3x - 8$ and $f_2(x) = 7x^2 + 9x + 7$.

Solution: The crossing points are exact those points of both curves, where the x- and the y-values are identic. Therefore we can equate both functional terms to get them.

$$\begin{aligned}
4x_S^2 + 3x_S - 8 &= 7x_S^2 + 9x_S + 7 && | - 7x_S^2 - 9x_S - 7 \\
-3x_S^2 - 6x_S - 15 &= 0 && | : (-3) \\
x_S^2 + 2x_S + 5 &= 0 \\
x_{S1/2} &= -1 \pm \sqrt{1-5} \\
x_{S1/2} &= -1 \pm \sqrt{-4}
\end{aligned}$$

As there is no real solution for the root, we have **no** crossing points.

You see the course of the funktions underneath.



5.10 Exercise 10:

Find out the functional equation of the parabula, that crosses these points: $P_1(-1|8)$, $P_2(2|-1)$ und $P_3(4|3)$

Solution: To solve the problem we start with the normal form of the quadratic function. When we insert in each case the coordinates of all known points, we get three equations als a system of linear equations. With this we can figure out the parameters a, b and c. The normal form is:

$$f(x) = ax^2 + bx + c$$

We insert the coordinates of the three points.

$$\begin{array}{rcrcrcrcr} P_1(-1|8) & \Rightarrow & f(-1) & = & 8 & \Rightarrow & a\cdot(-1)^2 + b\cdot(-1) + c & = & 8 \\ P_2(2|-1) & \Rightarrow & f(2) & = & -1 & \Rightarrow & a\cdot 2^2 + b\cdot 2 + c & = & -1 \\ P_3(4|3) & \Rightarrow & f(4) & = & 3 & \Rightarrow & a\cdot 4^2 + b\cdot 4 + c & = & 3 \end{array}$$

On the right side we got the system of linear equations. We simplify it a little bit:

$$\begin{array}{lrrrcr} (1) & a & -b & +c & = & 8 \\ (2) & 4a & +2b & +c & = & -1 \\ (3) & 16a & +4b & +c & = & 3 \end{array}$$

This system of linear equations can be solved with any known methods, for example the method of insertion, the method of addition, the rule of Cramer or the method of Gauß and Jordan. I prefer the rule of Cramer[11]. I start to calculate a:

$$a = \frac{\begin{vmatrix} 8 & -1 & 1 \\ -1 & 2 & 1 \\ 3 & 4 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{vmatrix}} = \frac{16-3-4-6-32-1}{2-16+16-32-4+4} = \frac{-30}{-30} = 1$$

With the now known value of the determinant in the denominator we quickly get b:

$$b = \frac{\begin{vmatrix} 1 & 8 & 1 \\ 4 & -1 & 1 \\ 16 & 3 & 1 \end{vmatrix}}{-30} = \frac{-1+128+12+16-3-32}{-30} = \frac{120}{-30} = -4$$

The easyest way to get c is insertion of the known values a and b into any equation. I take equation (1).

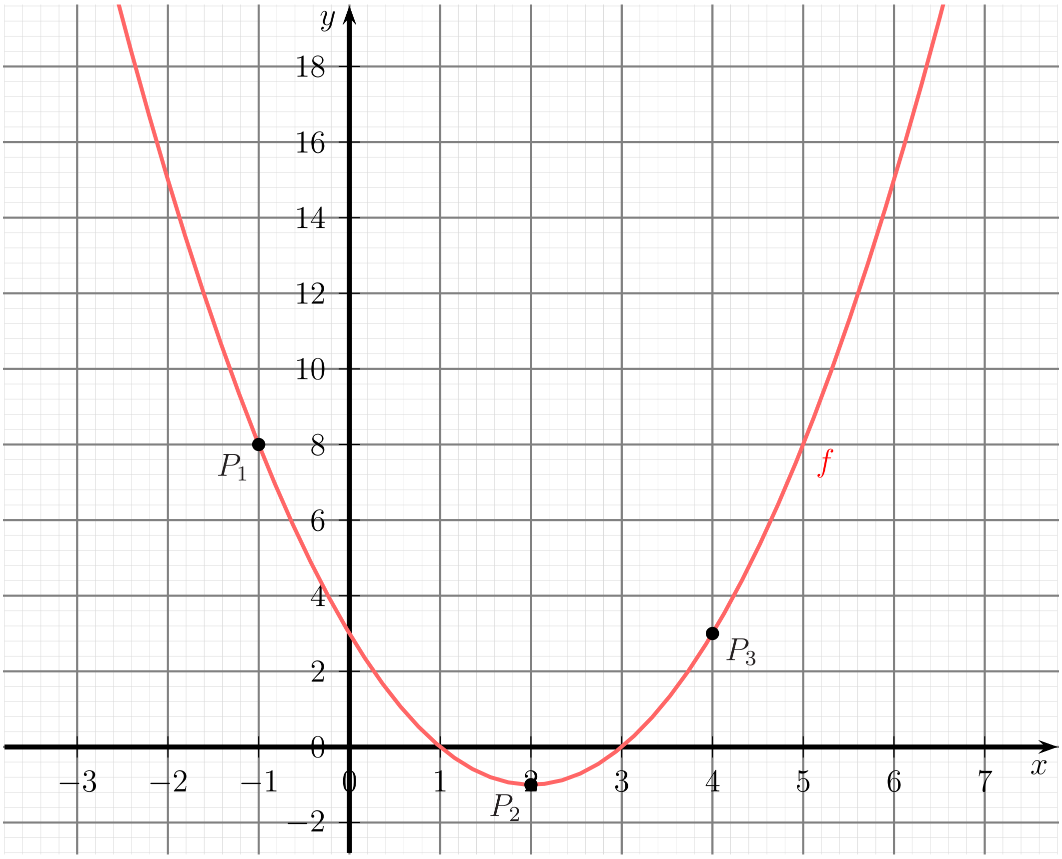
$$\begin{array}{rcll} a-b+c & = & 8 & \\ 1-(-4)+c & = & 8 & \\ 5+c & = & 8 & |-5 \\ c & = & 3 & \end{array}$$

Inserting the calculates values into the normal form we get the wanted equation.

$$\boxed{f(x) = x^2 - 4x + 3}$$

[11] For details look here: http://www.dk4ek.de/lib/exe/fetch.php/cramer.pdf

You see the course of the funktion underneath.



5.11 Exercise 11:

Find out the functional equation of the linear function $f_1(x)$, whose straight line touches the parabula with the functional equation $f_1(x) = x^2 - 4x + 4$ at $x_b = 4$ as a tangent.

Solution: Let us start with the normal form of the linear function. It looks li'ke this:

$$f_2(x) = mx + b$$

The concept of solving the problem is as follows: A straight line **touches** a parabula, if and only if there ist **only one** common point. This is the touching point, not two crossing points.

We can start in the same way like looking for crossing points. The x-value of the "crossing point" is called x_t then.

$$\begin{aligned}
f_1(x_t) &= f_2(x_t) \\
x_t^2 - 4x_t + 4 &= m \cdot x_t + b && | - m \cdot x_t - b \\
x_t^2 - 4x_t - m \cdot x_t + 4 - b &= 0 \\
x_t^2 - (4+m) \cdot x_t + 4 - b &= 0 && | \text{ } p\text{-}q\text{-formula} \\
x_{t1/2} &= \frac{4+m}{2} \pm \sqrt{\left(\frac{4+m}{2}\right)^2 - 4 + b}
\end{aligned}$$

As told before we have a **touching point**, when we get **only one** common point. For this effect the root must have the value zero. In this case we get:

$$x_B = \frac{4+m}{2}$$

As we already know the value $x_B = 4$, we can calculate m by this.

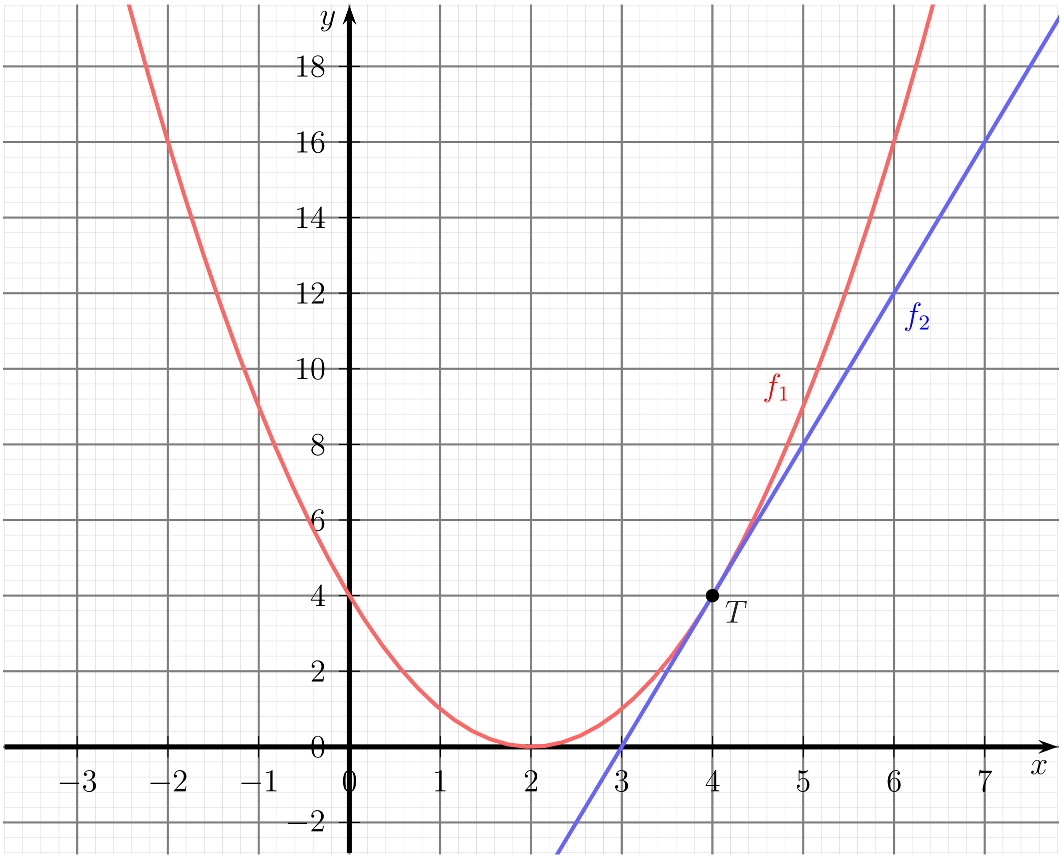
$$\begin{aligned}
x_B &= \frac{4+m}{2} \\
4 &= \frac{4+m}{2} && | \cdot 2 \\
8 &= 4 + m && | - 4 \\
4 &= m
\end{aligned}$$

We know, the content of the root is zero. Hereby we get an equation to work out b.

$$\begin{aligned}
\left(\frac{4+m}{2}\right)^2 - 4 + b &= 0 \\
\left(\frac{4+4}{2}\right)^2 - 4 + b &= 0 \\
16 - 4 + b &= 0 \\
12 + b &= 0 && | - 12 \\
b &= -12
\end{aligned}$$

We have got the wanted functional equation: $f_2(x) = 4x - 12$

Here we see the courses of the functional graphs.



5.12 Exercise 12:

A parabula has the vertex $V(3|2)$ and runs through the point $P(5|10)$. Find out its funktional equation $f(x)$!

Solution: As we know the vertex, it is appropriate, to use the vertex form. We only have to calculate then the shape factor a.

$$\begin{aligned} f(x) &= a \cdot (x - x_V)^2 + y_V \\ f(x) &= a \cdot (x-3)^2 + 2 \end{aligned}$$

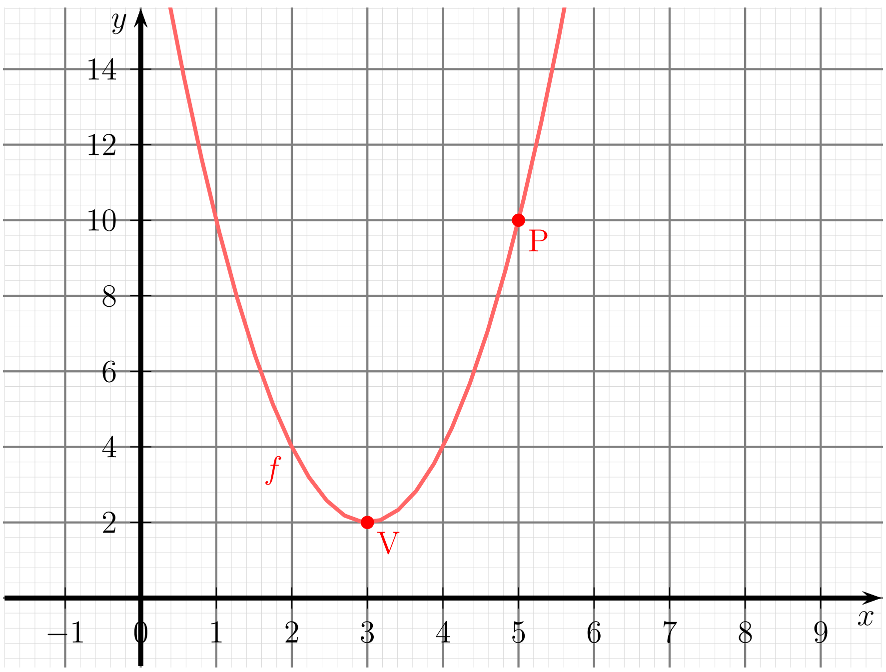
Furthermore we know the coordinates of the point P. They must comply the functional equation.

$$\begin{aligned} f(x_p) &= y_p \\ a \cdot (x_p - 3)^2 + 2 &= y_p \\ a \cdot (5-3)^2 + 2 &= 10 \\ a \cdot 2^2 + 2 &= 10 \quad | - 2 \\ a \cdot 4 &= 8 \quad\;\; | : 4 \\ a &= 2 \end{aligned}$$

I insert this value as a and I get the wanted functional equation.

$$\begin{aligned} f(x) &= 2 \cdot (x-3)^2 + 2 \\ &= 2 \cdot (x^2 - 6x + 9) + 2 \\ &= 2x^2 - 12x + 18 + 2 \\ f(x) &= 2x^2 - 12x + 20 \end{aligned}$$

Here we see the course of the functional graph.



5.13 Exercise 13:

Calculate the zeros and the vertex of the quadratic function (as far they exist):

$$f(x) = -16x^2 + 24x - 25$$

Solution: At a zero the value of the function is zero. That leads us to the base of solving the problem: $f(x_0) = 0$.

$$\begin{aligned}
-16x_0^2 + 24x_0 - 25 &= 0 \qquad\qquad | : (-16) \\
x_0^2 - \frac{3}{2}x_0 + \frac{25}{16} &= 0 \\
x_{01/02} &= \frac{3}{4} \pm \sqrt{\frac{9}{16} - \frac{25}{16}} \\
x_{01/02} &= \frac{3}{4} \pm \sqrt{-1}
\end{aligned}$$

As there is no real solution for the root we have **no zeros**. On the other hand we can read out from this formula the x-value x_V of the vertex. It is always the value, we find in front of $\pm$ and the root. That is $x_V = \frac{3}{4}$. The corresponding y-value y_V we get by inserting x_V into the functional equation.

$$\begin{aligned}
y_V &= f(x_V) \\
&= -16x_S^2 + 24x_S - 25 \\
&= -16 \cdot \left(\frac{3}{4}\right)^2 + 24 \cdot \frac{3}{4} - 25 \\
&= -9 + 18 - 25 \\
y_V &= 16
\end{aligned}$$

Result: vertex $V\left(\frac{3}{4}\middle|16\right)$

5.14 Exercise 14:

The quadratic function $f(x)$ has the vertex $S(4|3)$ and runs through the point $P(6|11)$. Find out the functional equation of $f(c)$!

Solution: It is appropriate to use the vertex formula of the quadratic function.

$$\begin{aligned} f(x) &= a \cdot (x - x_S)^2 + y_S \\ f(x) &= a \cdot (x-4)^2 + 3 \end{aligned}$$

To get the value of the missing parameter a we insert the coordinates of the point P into this equation.

$$\begin{aligned} f(x_P) &= y_P \\ f(6) &= 11 \\ a \cdot (6-4)^2 + 3 &= 11 \\ a \cdot 4 + 3 &= 11 \quad | -3 \\ 4a &= 8 \quad | : 4 \\ a &= 2 \end{aligned}$$

If you want, yo can transform the vertex form into the normal form.

$$\begin{aligned} f(x) &= 2 \cdot (x-4)^2 + 3 \\ &= 2 \cdot (x^2 - 8x + 16) + 3 \\ &= 2x^2 - 16x + 32 + 3 \\ f(x) &= 2x^2 - 16x + 35 \end{aligned}$$

The wanted function is:

$$\boxed{f(x) = 2 \cdot (x-4)^2 + 3} \quad \text{or} \quad \boxed{f(x) = 2x^2 - 16x + 35}$$

5.15 Exercise 15:

Find out the functional equation of the parabula, that crosses these points: $P_1(-3|32)$, $P_2(-1|10)$ and $P_3(2|7)$

Solution: We start with the normal form $f(x) = ax^2 + bx + c$ and insert the coordinates of all points.

$$\begin{array}{lrcll} (1) & f(-3) & = & 32 & \Rightarrow & a\cdot(-3)^2 + b\cdot(-3) + c & = & 32 \\ (2) & f(-1) & = & 10 & \Rightarrow & a\cdot(-1)^2 + b\cdot(-1) + c & = & 10 \\ (3) & f(2) & = & 7 & \Rightarrow & a\cdot 2^2 + b\cdot 2 + c & = & 7 \end{array}$$

On the right side we have got the system of linear equations. We simplify it a little bit:

$$\begin{array}{lrrrcl} (1) & 9a & -3b & +c & = & 32 \\ (2) & a & -b & +c & = & 10 \\ (3) & 4a & +2b & +c & = & 7 \end{array}$$

This system of linear equations can be solved with any known methods, for example the method of insertion, the method of addition, the rule of Cramer or the method of Gauß and Jordan. I prefer the rule of Cramer. I start to calculate a:

$$a = \frac{\begin{vmatrix} 32 & -3 & 1 \\ 10 & -1 & 1 \\ 7 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 9 & -3 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{vmatrix}} = \frac{-32 - 21 + 20 + 7 - 64 + 30}{-9 - 12 + 2 + 4 - 18 + 3} = \frac{-60}{-30} = 2$$

With the now known value of the determinant in the denominator we quickly get b:

$$b = \frac{\begin{vmatrix} 9 & 32 & 1 \\ 1 & 10 & 1 \\ 4 & 7 & 1 \end{vmatrix}}{-30} = \frac{90 + 128 + 7 - 40 - 63 - 32}{-30} = \frac{90}{-30} = -3$$

The easyest way to get c is insertion of the known values a and b into any equation. I take equation (2).

$$\begin{array}{rcll} a - b + c & = & 10 & \\ 2 + 3 + c & = & 10 & | -5 \\ c & = & 5 & \end{array}$$

We get the wanted function: $\boxed{f(x) = 2x^2 - 3x + 5}$

5.16 Exercise 16:

A shifted normal parabula (factor of shape $a = 1$) runs through the points $P_1(-1|-2)$ und $P_2(0|1)$. What is the associated functional equation $f(x)$?

Solution: By using the known shape factor $a = 1$ we get the normal form:

$$f(x) = x^2 + bx + c$$

The coordinates of the known points are inserted.

$$\begin{array}{lrcrcrcl} (1) & f(-1) & = & -2 & \Rightarrow & (-1)^2 + b \cdot (-1) + c & = & -2 \\ (2) & f(0) & = & 1 & \Rightarrow & 0^2 + b \cdot 0 + c & = & 1 \end{array}$$

From equation (2) we directly get:

$$c = 1$$

We insert this value into equation (1).

$$\begin{array}{rcll} (-1)^2 + b \cdot (-1) + 1 & = & -2 & \\ 1 - b + 1 & = & -2 & \mid -2 \\ -b & = & -4 & \mid \cdot (-1) \\ b & = & 4 & \end{array}$$

So we have got the wanted functional equation: $f(x) = x^2 + 4x + 1$

5.17 Exercise 17:

Find out the functional equation of the parabula, that crosses these points: $P_1(0|8)$, $P_2(1|3)$ and $P_3(2|0)$

Solution: We start with the normal form $f(x) = ax^2 + bx + c$ and insert the coordinates of all points.

$$\begin{array}{lrcllcl} (1) & f(0) & = & 8 & \Rightarrow & a \cdot 0^2 + b \cdot 0 + c & = & 8 \\ (2) & f(1) & = & 3 & \Rightarrow & a \cdot 1^2 + b \cdot 1 + c & = & 3 \\ (3) & f(2) & = & 0 & \Rightarrow & a \cdot 2^2 + b \cdot 2 + c & = & 0 \end{array}$$

On the right side we have got the system of linear equations. We simplify it a little bit:

$$\begin{array}{lrrrcl} (1) & & & c & = & 8 \\ (2) & a & +b & +c & = & 3 \\ (3) & 4a & +2b & +c & = & 0 \end{array}$$

$c = 8$ bekannt ist. Dieser Wert wird vorab in (2) und (3) eingesetzt, die Gleichungen werden vereinfacht. This system of linear equations can be solved with any known methods. You should notice, that equation (1) directly tells us the value $c = 8$. We insert this value into equations (2) and (3). The generated equations will be simplified a bit.

$$\begin{array}{lrcll} (2) & a + b + 8 & = & 3 & | - 8 \\ (2) & a + b & = & -5 & \\ \hline (3) & 4a + 2b + 8 & = & 0 & | - 8 \\ (3) & 4a + 2b & = & -8 & \end{array}$$

In sum we get a system of linear equations of 2^{nd} order.

$$\boxed{\begin{array}{lrrcr} (2) & a & +b & = & -5 \\ (3) & 4a & +2b & = & -8 \end{array}}$$

Equation (2) easily can be equated to a or b, to use the method of insertion[12]. I equate to a.

$$\begin{array}{rcll} a + b & = & -5 & | - b \\ a & = & -5 - b & \end{array}$$

Insert into (3):

$$\begin{array}{rcll} 4a + 2b & = & -8 & \\ 4 \cdot (-5 - b) + 2b & = & -8 & \\ -20 - 4b + 2b & = & -8 & | + 20 \\ -2b & = & 12 & | : (-2) \\ b & = & -6 & \end{array}$$

[12]For details look here: http://www.dk4ek.de/lib/exe/fetch.php/einsetz.pdf

By using the transformed equation (2) we directly get a.

$$a = -5 - b = -5 - (-6) = -5 + 6 = 1$$

Inserting all results we get the wanted function: $\boxed{f(x) = x^2 - 6x + 8}$

5.18 Exercise 18:

A parabula (quadratic function) has the vertex $S(-2|0)$ and runs through the point $P(-1|-2)$. Specify the functional equation $f(x)$!

Solution: As we know the vertex, we should use the vertex form as a method of resolution. Only parameter $\boldsymbol{a}$ is missing.

$$\begin{aligned} f(x) &= a \cdot (x - x_V)^2 + y_V \\ f(x) &= a \cdot (x + 2)^2 + 0 \end{aligned}$$

Furthermore we know the coordinates of the point P. They must comply the functional equation.

$$\begin{aligned} f(x_p) &= y_p \\ a \cdot (x_p + 2)^2 &= y_p \\ a \cdot (-1 + 2)^2 &= -2 \\ a \cdot 1^2 &= -2 \\ a &= -2 \end{aligned}$$

Using this value we can show the funtional equation in vertex form.

$$f(x) = -2 \cdot (x + 2)^2$$

If you want, yo can transform the vertex form into the normal form.

$$f(x) = -2x^2 - 8x - 8$$

The wanted function is: $\boxed{f(x) = -2 \cdot (x + 2)^2}$ or $\boxed{f(x) = -2x^2 - 8x - 8}$

5.19 Exercise 19:

Take the parabula of the function $f_1(x) = \frac{1}{2}(x-4)^2 - 1$ and shift her so far **upwards**, that she will hit the point $P(2|3)$. What ist the functional equation $f_2(x)$ of the new function?

Solution: To shift a function verticaly, we only have to add a constant value. I name it k.

$$f_2(x) = \frac{1}{2}(x-4)^2 - 1 + k$$

In order that point P is expected to be hit by the parabula, the coordinates of P must fulfil the functional equation.

$$\begin{aligned}
f_2(x_P) &= y_P \\
f_2(2) &= 3 \\
\frac{1}{2}\cdot(2-4)^2 - 1 + k &= 3 \\
\frac{1}{2}\cdot(-2)^2 - 1 + k &= 3 \\
\frac{1}{2}\cdot 4 - 1 + k &= 3 \\
2 - 1 + k &= 3 \quad |-1 \\
k &= 2
\end{aligned}$$

This value is inserted as k in the functional equation.

$$f_2(x) = \frac{1}{2}(x-4)^2 - 1 + 2 = \frac{1}{2}(x-4)^2 + 1$$

If you want, yo can transform the vertex form into the normal form:

$$\begin{aligned}
f_2(x) &= = \tfrac{1}{2}(x-4)^2 + 1 \\
&= \frac{1}{2}\cdot(x^2 - 8x + 16) + 1 \\
&= \frac{1}{2}x^2 - 4x + 8 + 1 \\
f_2(x) &= \frac{1}{2}x^2 - 4x + 9
\end{aligned}$$

The expected functional equation is:

$\boxed{f_2(x) = \frac{1}{2}(x-4)^2 + 1}$ or: $\boxed{f_2(x) = \frac{1}{2}x^2 - 4x + 9}$

5.20 Exercise 20:

You should shift the parabula of the quadratic equation $f_1(x) = \frac{1}{2}x^2 - 4x + 7$ so far to the **left**, that the new parabula hits the point $P(0|\frac{7}{2})$. Specify the corresponding functional equation $f_2(x)$. Show **all** possible solutions!

Solution: A shifting to the **right** with the value k means, that we replace the variable x with $(x-k)$. Appropiately means a **negative** value for k a shifting to the **left**. If you want a **positive** value of k, you alternativly can use $(x+k)$ für a shifting to the **left**. In my way of solution I use the first alternative.

$$\begin{aligned} f_2(x) &= f_1(x-k) \\ f_2(x) &= \frac{1}{2}\cdot(x-k)^2 - 4(x-k) + 7 \end{aligned}$$

Now we replace the coordinates of P for the variables x and y. So we can calculate k.

$$\begin{aligned} f_2(0) &= \frac{7}{2} \\ \frac{1}{2}\cdot(0-k)^2 - 4\cdot(0-k) + 7 &= \frac{7}{2} \\ \frac{1}{2}\cdot k^2 + 4k + 7 &= \frac{7}{2} \qquad |\cdot 2 \\ k^2 + 8k + 14 &= 7 \qquad |-7 \\ k^2 + 8k + 7 &= 0 \\ k_{1/2} &= -4 \pm \sqrt{4^2 - 7} \\ k_{1/2} &= -4 \pm 3 \\ k_1 = -1 \qquad & k_2 = -7 \end{aligned}$$

As both results are **negative**, we have a shifting to the **left** in **both** cases. So both solutions are valid.

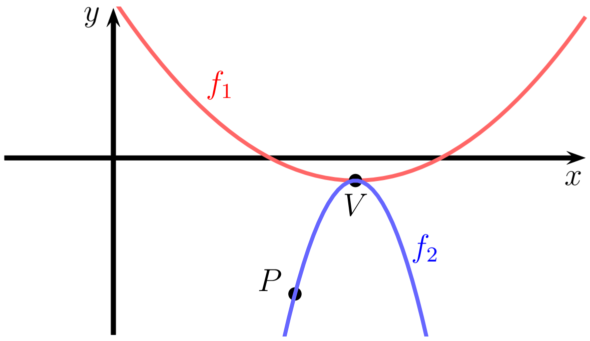
$$\begin{aligned} f_{21} &= \frac{1}{2}\cdot(x+1)^2 - 4\cdot(x+1) + 7 & f_{22} &= \frac{1}{2}\cdot(x+7)^2 - 4\cdot(x+7) + 7 \\ f_{21} &= \frac{1}{2}\cdot(x^2+2x+1) - 4x - 4 + 7 & f_{22} &= \frac{1}{2}\cdot(x^2+14x+49) - 4x - 28 + 7 \\ f_{21} &= \frac{1}{2}x^2 + x + \frac{1}{2} - 4x + 3 & f_{22} &= \frac{1}{2}x^2 + 7x + \frac{49}{2} - 4x - 21 \\ f_{21} &= \frac{1}{2}x^2 - 3x + \frac{7}{2} & f_{22} &= \frac{1}{2}x^2 + 3x + \frac{7}{2} \end{aligned}$$

Results: $f_{21} = \frac{1}{2}x^2 - 3x + \frac{7}{2}$ and: $f_{22} = \frac{1}{2}x^2 + 3x + \frac{7}{2}$

5.21 Exercise 21:

How has the parabula with the functional equation $f_1(x) = \frac{1}{2}x^2 - 4x + 7$ to be changed, that the point $P(3|-6)$ belongs to the new parabula? The new parabula should have the **same vertex** als the old one. Secify the functional equation $f_2(x)$ of the new parabula!

Solution: In order, that you can imagine the context better, you find here an outline of both functional graphs.



At first we have to find out the (common) vertex. The easiest way to do this, is using the vertex formula[13]:

$$f(x) = ax^2 + bx + c \qquad \Rightarrow \qquad x_V = -\frac{b}{2a}$$

$$x_V = -\frac{b}{2a} = -\frac{-4}{2 \cdot \frac{1}{2}} = 4$$

$$y_V = f(x_V) = \frac{1}{2} \cdot 4^2 - 4 \cdot 4 + 7 = -1$$

Using the known coordinates of the vertex we can set up the functional equation in vertex form.

$$f_2(x) = a \cdot (x - 4)^2 - 1$$

To determine parameter a we insert the coordinates of $P(3| - 6)$ into the functional equation.

$$\begin{aligned} f_2(3) &= -6 \\ a \cdot (3-4)^2 - 1 &= -6 \\ a \cdot (-1)^2 - 1 &= -6 \\ a - 1 &= -6 \quad |+1 \\ a &= -5 \end{aligned}$$

We can write down the functional equation:

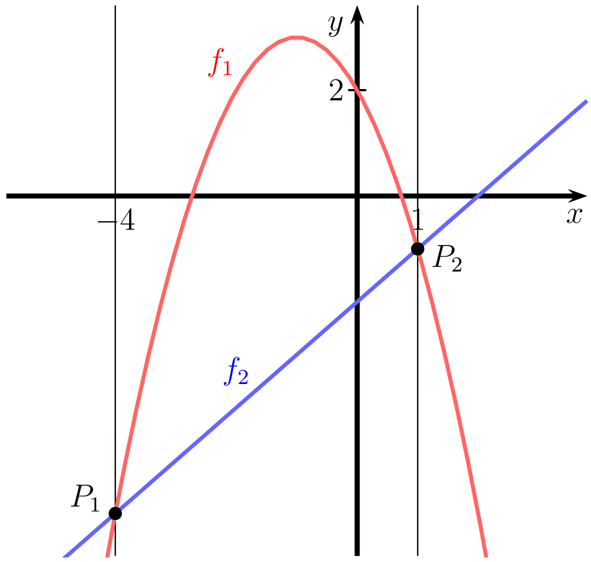
$$f_2(x) = -5 \cdot (x - 4)^2 - 1$$

[13]Look here in chapter 2.2

5.22 Exercise 22:

The parabula of the quadratic function $f_1(x)$ crosses the straight line of the function $f_2(x) = x - 2$ at $x_1 = -4$ and $x_2 = 1$. The parabula crosses the y-axis at $y_0 = 2$. Specify the functional equation $f_1(x)$!

Solution: In order, that you can imagine the context better, you find here an outline of both functional graphs.



At first we figure out the y-values of the crossing points P_1 and P_2. For that purpose we insert the x-values into the funktional equation f_2.

$$\begin{aligned} y_1 &= f_2(x_1) & \quad y_2 &= f_2(x_2) \\ y_1 &= -4 - 2 & \quad y_2 &= 1 - 2 \\ y_1 &= -6 & \quad y_2 &= -1 \end{aligned}$$

So we have got the crossing points $P_1(-4|-6)$ and $P_2(1|-1)$. The crossing point with the y-axis is located at $x = 0$. That means this point: $P_3(0|2)$ As we have **three** known points, we can set up a system of linear equations of the 3rd order, when we start with the normal form of a quadratic function:

$$f_2(x) = ax^2 + bx + c$$

$$\begin{array}{llcll} (1) & f(-4) = -6 & \Rightarrow & a \cdot (-4)^2 + b \cdot (-4) + c & = -6 \\ (2) & f(1) = -1 & \Rightarrow & a \cdot 1^2 + b \cdot 1 + c & = -1 \\ (3) & f(0) = 2 & \Rightarrow & a \cdot 0^2 + b \cdot 0 + c & = 2 \end{array}$$

When we condense the system of linear equations on the right side, we get:

$$\boxed{\begin{array}{lrrrcr} (1) & 16a & -4b & +c & = & -6 \\ (2) & a & +b & +c & = & -1 \\ (3) & & & c & = & 2 \end{array}}$$

As equation (3) immediately tells us the value $c = 3$, we can insert this value in both other equations. We get a system of linear equations of the 2^{nd} order.

$$\begin{array}{lrcl}
(1) & 16a - 4b + 2 & = & -6 \quad | - 2 \\
(2) & a + b + 2 & = & -1 \quad | - 2 \\
\hline
(1) & 16a - 4b & = & -8 \\
(2) & a + b & = & -3
\end{array}$$

This system of linear equations can be solved with an arbitrary way of solution. I want to use the method of insertion. Therefore I solve equation (2) to a and insert the solution into equation (1).

$$\begin{array}{lrcll}
(2) & a + b & = & -3 & | - b \\
 & a & = & -3 - b &
\end{array}$$

Inserted into (1):

$$\begin{array}{lrcll}
(1) & 16a - 4b & = & -8 & \\
 & 16 \cdot (-3 - b) - 4b & = & -8 & \\
 & -48 - 16b - 4b & = & -8 & | + 48 \\
 & -20b & = & 40 & | : (-20) \\
 & b & = & -2 &
\end{array}$$

This value is inserted into the transformed equation (2).

$$a = -3 - b = -3 - (-2) = -1$$

Now we know all parameters. We can write down the functional equation.

$$\boxed{f_2(x) = -x^2 - 2x + 2}$$

5.23 Exercise 23:

Calculate the crossing points – if they exist – of the two parabulas with the functional equations $f_1(x) = x^2 - 2x + 3$ and $f_2(x) = 2x^2 - 8x + 12$.

Solution: To find out the crossing points we equate the functional equations.

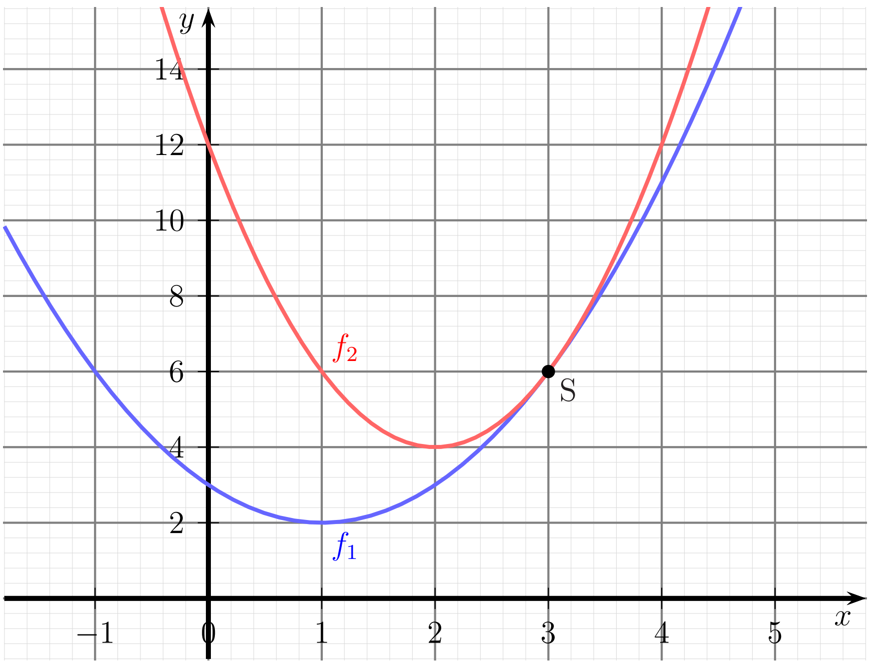
$$\begin{aligned} f_1(x_S) &= f_2(x_S) \\ x_S^2 - 2x_S + 3 &= 2x_S^2 - 8x_S + 12 \quad | - 2x_S^2 + 8x_S - 12 \\ -x_S^2 + 6x_S - 9 &= 0 \quad | \cdot (-1) \\ x_S^2 - 6x_S + 9 &= 0 \\ x_{S1/2} &= 3 \pm \sqrt{9 - 9} \\ x_S &= 3 \end{aligned}$$

We have got only **one** value. So we have only one crossing point, a **touching point**. To specify the y-value we insert this value x_S into f_1 or f_2. I want to use f_1.

$$\begin{aligned} y_S &= f_1(x_S) \\ &= x_S^2 - 2x_S + 3 \\ &= 3^2 - 2 \cdot 3 + 3 \\ y_S &= 6 \end{aligned}$$

We can show the crossing point: $S(3|6)$

You see the course of the funktions underneath.



5.24 Exercise 24:

A parabula with the shape factor $a = 2$ runs through the points $P_1(1|-1)$ and $P_2(2|1)$. Specify the functional equation $f(x)$!

Solution: We know the normal form of the quadratic function:

$$f(x) = ax^2 + bx + c$$

As a is known as $a = 2$, we can specify:

$$f(x) = 2x^2 + bx + c$$

Now we have to calculate the parameters b and c. For that purpose we insert the known coordinates of $P_1(1|-1)$ and $P_2(2|1)$ as x and y into this formula.

$$\begin{array}{llllll} P_1(1|-1) & \Rightarrow & f(1) = -1 & \Rightarrow & 2 \cdot 1^2 + b \cdot 1 + c & = & -1 \\ P_2(2|1) & \Rightarrow & f(2) = 1 & \Rightarrow & 2 \cdot 2^2 + b \cdot 2 + c & = & 1 \end{array}$$

Both equations at first become simplified and transformed into the normal form.

$$\begin{array}{lrcll} (1) & 2 + b + c & = & -1 & |-2 \\ (2) & 8 + 2b + c & = & 1 & |-8 \\ \hline (1) & b + c & = & -3 & \\ (2) & 2b + c & = & -7 & \end{array}$$

Now we have got a 2^{nd} order system of linear equations. We can solve it with any method. As we find parameter c in both equations without any coefficient, it is suggesting itself to subtract both equations from each other. Parameter c then will disappear. As the coefficient of b in equation (2) – here: $2b$ – is larger, as in equation (1) – here: $1b$ – I subtract equation (1) from equation (2). Then the result will be **positive**. (Of course it also will work otherwise, but then we will get $-b$.)

$$\begin{array}{rrrcrl} (1) & b & +c & = & -3 & |- \\ (2) & 2b & +c & = & -7 & | \\ \hline (2)-(1) & b & & = & -4 & \end{array}$$

Now we already have b. To get the remainig parameter c we put this result into one of the two equations. As in (1) the numbers are a little bit smaller, I use this equation.

$$\begin{array}{rcll} b + c & = & -3 & \\ -4 + c & = & -3 & |+4 \\ c & = & 1 & \end{array}$$

With this results we can write down the functional equation:

$$\boxed{f(x) = 2x^2 - 4x + 1}$$

5.25 Exercise 25:

A parabula crosses the points $P_1(2|7)$ und $P_2(3|6)$ and cuts the y-axis at $y_0 = 3$. Find out the functional equation $f(x)$!

Solution: We start with the normal form:

$$f(x) = ax^2 + bx + c$$

The y-axis intercept at $y_0 = 3$ means $c = y_0 = 3$. With that the equation gets this form:

$$f(x) = ax^2 + bx + 3$$

We now have to calculate the parameters a and b. For that purpose we place the coordinates of the known points into the functional equation as x and y.

$$\begin{array}{lllllll} P_1(2|7) & \Rightarrow & f(2) = 7 & \Rightarrow & a \cdot 2^2 + b \cdot 2 + 3 & = & 7 \\ P_2(3|6) & \Rightarrow & f(3) = 6 & \Rightarrow & a \cdot 3^2 + b \cdot 3 + 3 & = & 6 \end{array}$$

Both resulting equations become simplified an then transformed into the normal form.

$$\begin{array}{lrcll} (1) & 4a + 2b + 3 & = & 7 & | -3 \\ (2) & 9a + 3b + 3 & = & 6 & | -3 \\ \hline (1) & 4a + 2b & = & 4 & \\ (2) & 9a + 3b & = & 3 & \end{array}$$

We have got a system of linear equations of the 2^{nd} order to solve within an arbitrary method. I use the **method of insertion**, as no method is particulary suitable. Therefore I transform equation (1) to b.

$$\begin{array}{lrcll} (1) & 4a + 2b & = & 4 & | -4a \\ & 2b & = & 4 - 4a & | : 2 \\ & b & = & 2 - 2a & \end{array}$$

The result is inserted into equation (2).

$$\begin{array}{lrcll} (2) & 9a + 3b & = & 3 & \\ & 9a + 3 \cdot (2 - 2a) & = & 3 & \\ & 9a + 6 - 6a & = & 3 & | -6 \\ & 3a & = & -3 & | : 3 \\ & a & = & -1 & \end{array}$$

To get the last unknown parameter b we insert this result into the transformed equation (1).

$$b = 2 - 2a = 2 - 2 \cdot (-1) = 2 + 2 = 4$$

So we have got the wanted functional equation:

$$\boxed{f(x) = -x^2 + 4x + 3}$$

5.26 Exercise 26:

Shift the parabula with the functional equation $f_1(x) = 3x^2 - 4$ in a way, that the new parabula crosses the points $P_1(3|2)$ und $P_2(6|-7)$. Specify the functional equation $f_2(x)$, that occured by this shifting.

Solution: The only detail, that does not change by shifting, is the **shape** of the parabula and with it the shape factor a. We can read the shape factor a from f_1 as $a = 3$. With this value we get the normal form of the functional equation:

$$f_2(x) = 3x^2 + bx + c$$

We place the coordinates of the known points into the functional equation as x and y.

$$\begin{array}{llllll} P_1(3|2) & \Rightarrow & f(3) = 2 & \Rightarrow & 3 \cdot 3^2 + b \cdot 3 + c & = & 2 \\ P_2(6|-7) & \Rightarrow & f(6) = -7 & \Rightarrow & 3 \cdot 6^2 + b \cdot 6 + c & = & -7 \end{array}$$

Both resulting equations become simplified an then transformed into the normal form.

$$\begin{array}{lrcll} (1) & 27 + 3b + c & = & 2 & |-27 \\ (2) & 108 + 6b + c & = & -7 & |-108 \\ \hline (1) & 3b + c & = & -25 & \\ (2) & 6b + c & = & -115 & \end{array}$$

Now we have got a 2^{nd} order system of linear equations. We can solve it with any method. As we find parameter c in both equations without any coefficient, it is suggesting itself to subtract both equations from each other. Parameter c then will disappear. As the coefficient of b in equation (2) – here: $6b$ – is larger, as in equation (1) – here: $1b$ – I subtract equation (1) from equation (2). Then the result will be **positive**. (Of course it also will work otherwise, but then we will get $-b$.)

$$\begin{array}{rllcll} (1) & 3b & +c & = & -25 & |- \\ (2) & 6b & +c & = & -115 & | \\ \hline (2)-(1) & 3b & & = & -90 & |:3 \\ & b & & = & -30 & \end{array}$$

Now we already have b. To get the remainig parameter c we put this result into one of the two equations. As in (1) the numbers are a little bit smaller, I use this equation.

$$\begin{array}{rcll} 3b + c & = & -25 & \\ 3 \cdot (-30) + c & = & -25 & \\ -90 + c & = & -25 & |+90 \\ c & = & 65 & \end{array}$$

With this results we can write down the wanted functional equation:

$$\boxed{f(x) = 3x^2 - 30x + 65}$$

5.27 Exercise 27:

A quadratic function has the shape factor $a = -1$ and the vertex $V(2|3)$. Specify the functional equation $f(x)$ in **vertex form and in normal form**!

Solution: We immediately can write down the functional equation in vertex form:

$$\boxed{f(x) = -(x-2)^2 + 3}$$

The transformation into the normal form can be done by using the 2^{nd} binomial formula.

$$\begin{aligned} f(x) &= -(x-2)^2 + 3 \\ &= -(x^2 - 4x + 4) + 3 \\ &= -x^2 + 4x - 4 + 3 \\ f(x) &= -x^2 - 4x - 1 \end{aligned}$$

Here we have the functional equation in normal form:

$$\boxed{f(x) = -x^2 - 4x - 1}$$

5.28 Exercise 28:

The functional graph of a quadratic function has the vertex $V(3|-1)$ and runs through the point $P(1|7)$. Specify the functional equation $f(x)$ in **vertex form and in normal form**!

Solution: We start with the concept of vertex form.

$$f(x) = a \cdot (x-3)^2 - 1$$

To figure out the shape factor a we insert the coordinates of the known point $P(1|7)$.

$$\begin{aligned} f(x_p) &= y_p \\ a \cdot (x_p - 3)^2 - 1 &= y_p \\ a \cdot (1-3)^2 - 1 &= 7 \\ a \cdot (-2)^2 - 1 &= 7 \\ 4a - 1 &= 7 \quad |+1 \\ 4a &= 8 \quad |:4 \\ a &= 2 \end{aligned}$$

With this we can write down the functional equation in vertex form.

$$\boxed{f(x) = 2 \cdot (x-3)^2 - 1}$$

The transformation into the normal form can be done by using the 2^{nd} binomial formula.

$$\begin{aligned} f(x) &= 2 \cdot (x-3)^2 - 1 \\ &= 2 \cdot (x^2 - 6x + 9) - 1 \\ &= 2x^2 - 12x + 18 - 1 \\ f(x) &= 2x^2 - 12x + 17 \end{aligned}$$

Here we have the functional equation in normal form:

$$\boxed{f(x) = 2x^2 - 12x + 17}$$

5.29 Exercise 29:

We have a quadratic function with the functional equation $f(x) = -2x^2 - 4x + 6$. Calculate the vertex and the zeros!

Solution:

First variant of solution: At first we calculate the zeros. "On the way" we can find the x-value x_V of the vertex.

To figure out the zeros we must equate the functional term to zero.

$$\begin{aligned} -2x_0^2 - 4x_0 + 6 &= 0 && \mid : (-2) \\ x_0^2 + 2x_0 - 3 &= 0 && \mid \text{use } p\text{-}q\text{-formula} \\ x_{01/02} &= -\tfrac{2}{2} \pm \sqrt{\left(\tfrac{2}{2}\right)^2 - (-3)} \\ &= -1 \pm \sqrt{1+3} \\ &= -1 \pm 2 \\ x_01 = -1 + 2 = 1 & \quad x_02 = -1 - 2 = -3 \end{aligned}$$

The zeros ar: $\boxed{x_{01} = 1}$ and $\boxed{x_{02} = -3}$

In front of the sign $\pm$ and the root we find the value -1. This is the x-value x_V of the vertex.

$$x_V = -1$$

To find the corresponding y-Value y_V we insert $x_V = -1$ into the functional equation.

$$\begin{aligned} y_V &= f(x_V) \\ &= -2x_V^2 - 4x_V + 6 \\ &= -2 \cdot (-1)^2 - 4 \cdot (-1) + 6 \\ &= -2 + 4 + 6 \\ y_V &= 8 \end{aligned}$$

The vertex is named: $\boxed{S(-1|8)}$

Second variant of solution: We can find the x-value x_V of the vertex with the vertex formula.

$$\begin{aligned} x_V &= -\frac{b}{2a} \\ &= -\frac{-4}{2 \cdot (-2)} \\ &= \frac{4}{-4} \\ x_V &= -1 \end{aligned}$$

(Continue like described in first variant of solution.)

5.30 Exercise 30:

A parabula with the vertex $V(4|7)$ crosses the y-axis at $y_0 = 4$. Specify the associated functional equation!

Solution: We start with the vertex form of the functional equation.

$$f(x) = a \cdot (x-4)^2 + 7$$

To calculate the missing shape factor a, we have two possibilities.

First variant of solution: The y-axis is placed at the position $x = 0$. So we know the crossing point at $(0|y_0)$ respectively $(0|3)$. We insert this coordinates into the vertex form.

$$\begin{aligned} f(0) &= 3 \\ a \cdot (0-4)^2 + 7 &= 3 \\ a \cdot 16 + 7 &= 3 \qquad | -7 \\ 16a &= -4 \qquad | : 16 \\ a &= -\frac{1}{4} \end{aligned}$$

We have got the functional equation in vertex form:

$$\boxed{f(x) = -\tfrac{1}{4} \cdot (x-4)^2 + 7}$$

Second variant of solution: The other procedure to get a is as follows. We start with the vertex form, transform it into the normal form and read out the parameter c. This value is the value of y_0.

$$\begin{aligned} f(x) &= a \cdot (x-4)^2 + 7 \\ &= a \cdot (x^2 - 8x + 16) + 7 \\ f(x) &= ax^2 - 8ax + \underbrace{16a + 7}_{c} \end{aligned}$$

We equate c with the known value $y_0 = 3$.

$$\begin{aligned} c &= y_0 \\ 16a + 7 &= 3 \qquad | -7 \\ 16a &= -4 \qquad | : 16 \\ a &= -\frac{1}{4} \end{aligned}$$

By this variant of solution we get the same functional equation in vertex form:

$$\boxed{f(x) = -\tfrac{1}{4} \cdot (x-4)^2 + 7}$$

5.31 Exercise 31:

We have a quadratic function with the functional equation:

$$f_1(x) = 3x^2 + 24x + 53$$

We are looking for the function f_2 with the same vertex as f_1. The related parabula of f_2 should cross the point $P(-2|-3)$.

Solution: First we have to find out the vertex of f_1. We use the vertex formula.

$$\begin{aligned} x_V &= -\frac{b}{2a} \\ &= -\frac{24}{2 \cdot 3} \\ x_V &= -4 \end{aligned}$$

The corresponding y-value y_V we get by using the functional equation of f_1.

$$\begin{aligned} y_V &= f_1(x_V) \\ &= 3x_V^2 + 24x_V + 53 \\ &= 3 \cdot (-4)^2 + 24 \cdot (-4) + 53 \\ &= 48 - 96 + 53 \\ y_V &= 5 \end{aligned}$$

So the common vertex of f_1 and f_2 is named: $\boxed{S(-4|5)}$

With this results we can setup the functional equation of f_2 in vertex form.

$$f_2(x) = a \cdot (x+4)^2 + 5$$

To figure out the shape factor a we insert the coordinates of the known point $P(-2|-3)$.

$$\begin{aligned} f_2(-2) &= -3 \\ a \cdot (-2+4)^2 + 5 &= -3 \quad | -5 \\ a \cdot 2^2 &= -8 \\ 4a &= -8 \quad | : 4 \\ a &= -2 \end{aligned}$$

With this result we can write down the functional equation of f_2:

$$\boxed{f(x) = -2 \cdot (x+4)^2 + 5}$$

5.32 Exercise 32:

A parabula crosses the points $P_1(0|14)$ and $P_2(2|-2)$. The vertex can be found at $x_V = 3$. Specify the related functional equation $f(x)$!

Solution: Using the vertex $V(3|y_V)$ we can make the solution approach with the functional equation in vertex form.

$$f(x) = a \cdot (x-3)^2 + y_V$$

We insert the coordinates of both points $P_1(0|14)$ and $P_2(2|-2)$.

$$\begin{array}{rcrcrcl} P_1(0|14) & \Rightarrow & f(0) = 14 & \Rightarrow & a \cdot (0-3)^2 + y_V & = & 14 \\ P_2(2|-2) & \Rightarrow & f(2) = -2 & \Rightarrow & a \cdot (2-3)^2 + y_V & = & -2 \end{array}$$

We have got a system of linear equations of 2nd order with the variables a and b. We simplify the equations a little bit.

$$\begin{array}{lrcl} (1) & a \cdot (0-3)^2 + y_V & = & 14 \\ (2) & a \cdot (2-3)^2 + y_V & = & -2 \\ \hline (1) & 9a + y_V & = & 14 \\ (2) & a + y_V & = & -2 \end{array}$$

To solve this system of linear equations it is recommendable to use method of addition and subtraction[14]. We can subtract equation (2) from equation (1), then y_V will disappear.

$$\begin{array}{lrlcrl} (1) & 9a & +y_V & = & 14 & | \\ (2) & a & +y_V & = & -2 & |- \\ \hline & 8a & & = & 16 & | :8 \\ & a & & = & 2 & \end{array}$$

By the use of equation (2) we get y_V.

$$\begin{array}{rcll} a + y_V & = & -2 & \\ 2 + y_V & = & -2 & | -2 \\ y_V & = & -4 & \end{array}$$

Now all parameters are known. We can write down the functional equation:

$$\boxed{f(x) = 2 \cdot (x-3)^2 - 4}$$

If you want, you can transform this equation into normal form.

$$\begin{array}{rcl} f(x) & = & 2 \cdot (x-3)^2 - 4 \\ & = & 2 \cdot (x^2 - 6x + 9) - 4 \\ & = & 2x^2 - 12x + 18 - 4 \\ f(x) & = & 2x^2 - 12x + 14 \end{array}$$

[14] For details of the method of addition and subtraction look here:
http://www.dk4ek.de/lib/exe/fetch.php/add.pdf

5.33 Exercise 33:

Which of these five functions have the same vertex?

$$\begin{aligned} f_1(x) &= x^2 - 10x + 21 \\ f_2(x) &= 3x^2 - 30x + 71 \\ f_3(x) &= -2x^2 + 20x - 48 \\ f_4(x) &= 0{,}5x^2 + 5x + 8{,}5 \\ f_5(x) &= 4 \cdot (x-5)^2 - 4 \end{aligned}$$

Solution: To answer this question we must figure out all vertexes.

$$\begin{aligned} x_{V1} &= -\frac{b}{2a} \\ &= -\frac{-10}{2 \cdot 1} \\ x_{V1} &= 5 \end{aligned}$$

$$\begin{aligned} y_{V1} &= f_1(x_{V1}) \\ &= x_{V1}^2 - 10x_{V1} + 21 \\ &= 5^2 - 10 \cdot 5 + 21 \\ y_{V1} &= -4 \end{aligned}$$

$$\boxed{V_1(5|-4)}$$

$$\begin{aligned} x_{V2} &= -\frac{b}{2a} \\ &= -\frac{-30}{2 \cdot 3} \\ x_{V2} &= 5 \end{aligned}$$

$$\begin{aligned} y_{V2} &= f_2(x_{V2}) \\ &= 3x_{V2}^2 - 30x_{V2} + 71 \\ &= 3 \cdot 5^2 - 30 \cdot 5 + 71 \\ &= 75 - 150 + 71 \\ y_{V2} &= -4 \end{aligned}$$

$$\boxed{V_2(5|-4)}$$

$$\begin{aligned} x_{V3} &= -\frac{b}{2a} \\ &= -\frac{20}{2 \cdot (-2)} \\ x_{V3} &= 5 \end{aligned}$$

$$\begin{aligned} y_{V3} &= f_3(x_{V3}) \\ &= -2 \cdot x_{V3}^2 + 20x_{V3} - 48 \\ &= -2 \cdot 5^2 + 20 \cdot 5 - 48 \\ &= -50 + 100 - 48 \\ y_{V3} &= 2 \end{aligned}$$

$$\boxed{V_3(5|2)}$$

$$\begin{aligned} x_{V4} &= -\frac{b}{2a} \\ &= -\frac{5}{2 \cdot 0{,}5} \\ x_{V4} &= -5 \end{aligned}$$

$$\begin{aligned} y_{V4} &= f_4(x_{V4}) \\ &= 0{,}5 \cdot x_{V4}^2 + 5x_{V4} + 8{,}5 \\ &= 0{,}5 \cdot (-5)^2 + 5 \cdot (-5) + 8{,}5 \\ &= 12{,}5 - 25 + 8{,}5 \\ y_{V4} &= -4 \end{aligned}$$

$$\boxed{V_4(-5|-4)}$$

As f_5 is given in vertex form, we immediately can read out the coordinates of the vertex.

$$\boxed{V_5(5|-4)}$$

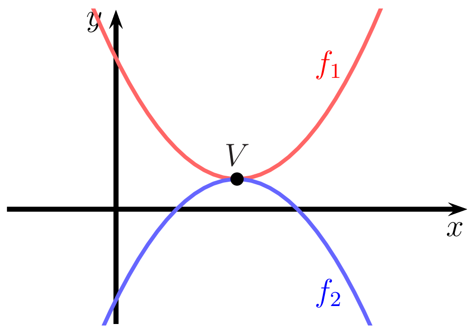
Result: The vertexes V_1, V_2 and V_5 are identically. At V_3 we have an other y-value, at V_4 is an other x-value.

5.34 Exercise 34:

We have given the function f_1 with the functional equation

$$f_1(x) = 4x^2 - 24x + 37$$

We want to find out the functional equation f_2 of a parabula, that is produced by rotating the parabula of f_1 with an angle of 180° around the vertex, as shown in the graphic on the right side.



Solution: At first we find out the vertex of f_1 by usung the vertex formula.

$$\begin{aligned} x_V &= -\frac{b}{2a} \\ &= -\frac{-24}{2 \cdot 4} \\ x_V &= 3 \end{aligned}$$

$$\begin{aligned} y_V &= f_1(x_V) \\ &= 4x_V^2 - 24x_V + 37 \\ &= 4 \cdot 3^2 - 24 \cdot 3 + 37 \\ &= 36 - 72 + 37 \\ y_V &= 1 \end{aligned}$$

$$\boxed{V(3|1)}$$

By rotaning the parabula with an angle of 180° the shape factor of $a_1 = 4$ is inverted.

$$\begin{aligned} a_2 &= -a_1 \\ a_2 &= -4 \end{aligned}$$

Using this data we can write down the functional equation in vertex form.

$$\boxed{f_2(x) = -4 \cdot (x-3)^2 + 1}$$

If you want, you can transform this into the normal form, although this is not requested.

$$\begin{aligned} f_2(x) &= -4 \cdot (x-3)^2 + 1 \\ &= -4 \cdot (x^2 - 6x + 9) + 1 \\ f_2(x) &= -4x^4 + 24x - 35 \end{aligned}$$

5.35 Exercise 35:

Find out the crossing points of the parabula with $f_1(x) = 10x^2 + 12x - 4$ and the stright line with $f_2(x) = 9x - 3$.

Solution:

$$\begin{aligned}
f_1(x_S) &= f_2(x_S) \\
10x_S^2 + 12x_S - 4 &= 9x_S - 3 && \mid -9x_S + 3 \\
10x_S^2 + 3x_S - 1 &= 0 && \mid : 10 \\
x_S^2 + 0{,}3x_S - 0{,}1 &= 0 \\
x_{S1/2} &= -0{,}15 \pm \sqrt{0{,}15^2 + 0{,}1} \\
&= -0{,}15 \pm 0{,}35 \\
x_{S1} = -0{,}15 + 0{,}35 = 0{,}2 & \qquad x_{S2} = -0{,}15 - 0{,}35 = -0{,}5 \\
y_{S1} &= f_2(x_{S1}) \\
&= 9 \cdot 0{,}2 - 3 \\
y_{S1} &= -1{,}2 \\
y_{S2} &= f_2(x_{S2}) \\
&= 9 \cdot (-0{,}5) - 3 \\
y_{S2} &= -7{,}5
\end{aligned}$$

Result: $S_1(0{,}2| - 1{,}2)$ and $S_2(-0{,}5| - 7{,}5)$

An outline of the graphs of the functional equations: